Robinson's Shorter Course.

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A

COMPLETE

ALGEBRA,

DESIGNED FOR USE IN

SCHOOLS, ACADEMIES, AND COLLEGES.

BY

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PREFACE.

THE author of this treatise on Algebra has undertaken the difficult task of preparing a work complete in one volume, which shall be sufficiently thorough for classes in Colleges and Universities, and at the same time sufficiently elementary for classes in Common Schools and Academies. To accomplish this desirable end the work has been so arranged that certain chapters and parts of chapters may be omitted by classes pursuing an elementary course.

The aim has been: 1. To treat each subject in harmony with the present modes of mathematical thinking; 2. To make every statement with such brevity and precision that the student cannot fail to understand the meaning; 3. To give a clear and rigorous demonstration of every proposition; 4. To present *one* difficulty at a time, and just at that stage of the student's progress when he is prepared to understand its treatment; 5. To treat with special care those subjects which have been found by experience to present peculiar difficulties; 6. To make the work thoroughly practical as well as thoroughly theoretical; 7. To present each subject in such a manner as to create a love for the study.

In the arrangement of subjects the author has departed widely from the beaten track; but he feels confident that the plan he has adopted will commend itself to the experienced and thoughtful teacher.

To facilitate frequent reviews, "Synopses for Review" have been placed at convenient intervals throughout the work.

To avoid making the present work too voluminous, Continued Fractions, Reciprocal Equations, Elimination by the Method of the Greatest Common Divisor, and Cardan's formula for cubic equations have been omitted. These subjects are treated in the Appendix to the author's "Book of Algebraic Problems."

In preparing the present treatise the author has first consulted his own experience as a teacher, and the book has been mainly written to meet the wants of his own classes; but he does not hesitate to acknowledge that he has received great assistance from many sources. A part of the material used in the chapters on Positive and Negative Quantities, Greatest Common Divisor and Least Common Multiple, Fractions, Simple Equations, Inequalities, Theory of Exponents, Mathematical Induction, and the sections on Permutations, Combinations, and Logarithms, has been taken from Prof. Todhunter's excellent treatise on Algebra. The works of Bertrand, Young, Peacock, Euler, Bland, Goodwin, and Wrigley have been consulted with advantage.

While the author has availed himself of such material in the books named as suited his purposes, it will be found that much of that so taken has long since become common property, having assumed a stereotyped form; and that other portions have been very much modified. It will be found, also, that the present treatise contains a large amount of new and original matter, which has not been inserted because it was novel, but because it served to simplify and elucidate the subject.

Special attention is called to the full and thorough manner in which the subject of Factoring is treated; to the demonstration of the Lemma, upon which the Binomial Theorem depends; to the classification and treatment of Radical Quantities; to the treatment of Quadratic Equations, Higher Equations, Simultaneous Equations, Ratio, Proportion, Progressions, Interpolation, Recurring Series, Reversion of Series; and to the Theory of Equations.

The chapter on "Logarithms and Expouential Equations" is almost entirely the work of Prof. James M. Greenwood, A. M., Superintendent of the Public Schools of Kansas City, Mo., and formerly Prof. of Math. in the North Missouri State Normal School; and the "Synopses for Review" have nearly all been prepared by Prof. George S. Bryant, A. M., of Christian College, Columbia, Mo. To these and other able and experienced teachers the author is also indebted for many valuable suggestions in relation to other portions of the work.

THE AUTHOR.

UNIVERSITY OF THE STATE OF MISSOURI,
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SUGGESTIONS TO TEACHERS.

1. If the examples and problems in the book are not sufficiently numerous or sufficiently varied, make some of your own, or take some from the Book of Examples and Problems, made to accompany this volume.

2. The Synopses for Review should be placed upon the black-board, and dwelt upon until the topics embraced in the review are thoroughly fixed in the mind of the student. To illustrate the manner of conducting a review, suppose the synopsis on page 10 is under consideration. Let the student point to the word "Algebra," and define it; then to "Algebraic Quantity," and define it; then to the two kinds of Algebraic Quantity—"Known and Unknown"—and define them; and so on.

3. The following chapters and parts of chapters may be omitted by classes pursuing an elementary course: That part of Chapter IV from Art. 125 to Art. 128 inclusive, and from Art. 133 to Art. 136 inclusive; Chapter VIII; Chapter XIV; that part of Chapter XVI from Art. 441 to Art. 457 inclusive; Chapter XVII; Arts. 482 and 483 of Chapter XVIII; all of Chapter XX after Geometrical Progressions; Chapter XXI; Chapter XXII; Chapter XXIII.

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ALGEBRA.

CHAPTER I.

DEFINITIONS AND NOTATION.

DEFINITIONS.

1. ***Algebra*** is that branch of Mathematics in which quantities are represented by letters, or by a combination of letters and figures, and in which the relations of quantities to each other and the operations to be performed are indicated by *Signs.*

The letters, figures, and signs are called *Symbols.*

2. ***Algebraic Language*** consists in the use of algebraic symbols.

3. ***An Algebraic Quantity*** or ***Expression*** is one expressed in algebraic language.

There are two kinds of algebraic quantities—*known* and *unknown.*

4. ***Known Quantities*** are those whose values are given. They are represented by numbers or the leading letters of the alphabet.

5. ***Unknown Quantities*** are those whose values are not given. They are represented by the final letters of the alphabet.

6. The sign $+$ is called the *plus sign,* and signifies that the quantity to which it is prefixed is to be *added.* Thus, $a + b$ signifies that b is to be added to a, and is read *a plus b.* If a represents 9, and b represents 3, then $a + b$ is equal to 12.

7. ***The Sum*** is the result obtained by addition.

8. The sign $-$ is called the *minus sign*, and signifies that the quantity to which it is prefixed is to be *subtracted*. Thus, $a-b$ signifies that b is to be subtracted from a, and is read *a minus b*. If a is 9, and b is 3, then $a-b$ is equal to 6.

9. ***The Remainder*** or ***Difference*** is the result obtained by subtraction.

10. The sign $\times$ is called the *sign of multiplication*, and signifies that the quantity which precedes it is to be *multiplied* by the one which follows it. Thus, $a \times b$ signifies that a is to be multiplied by b, and is read *a multiplied by b*, or *a into b*.

The sign of multiplication is often omitted. Thus, ab is equivalent to $a \times b$. Sometimes a point is used instead of the sign $\times$. Thus, $a \cdot b$ is equivalent to $a \times b$.

The sign of multiplication must not be omitted when the numbers are expressed by figures. Thus, 45 is not equivalent to 4×5.

11. ***The Product*** is the result obtained by multiplication.

12. The sign $\div$ is called the *sign of division*, and signifies that the quantity which precedes it is to be *divided* by the one which follows it. Thus, $a \div b$ signifies that a is to be divided by b, and is read *a divided by b*.

The expression $\frac{a}{b}$ is equivalent to $a \div b$.

13. ***The Quotient*** is the result obtained by division.

14. The sign $=$ is called the *sign of equality*, and signifies that the quantities between which it is placed are *equal*. Thus, $a=b$ signifies that a is equal to b, and is read *a equals b*, or *a is equal to b*.

15. ***An Equation*** consists of two expressions connected by the sign of equality. Thus, $x+y=a$, $a+b=c-d$, are equations.

The First Member of an equation is the quantity on the left of the sign of equality, and the *Second Member* is the quan-

tity on the right of the sign. Thus, in the equation, $x+y=a-b$, $x+y$ is the first member, and $a-b$ the second member.

16. The sign $>$ or $<$ is called the *sign of inequality,* and signifies that the quantities between which it is placed are *unequal,* the opening being turned toward the greater. Thus, $a>b$ signifies that a is greater than b, and is read *a is greater than b;* and $b<a$ signifies that b is less than a, and is read *b is less than a.*

17. ***An Inequality*** consists of two expressions connected by the sign of inequality, and its members are named as those of an equation.

18. When an expression is inclosed by a ***Parenthesis*** ( ), the operations which are indicated in that expression are to be regarded as performed, and the parenthesis is to be regarded as expressing the result. Thus, the expression $(a+b)(c-d)$ indicates that the sum of a and b is to be multiplied by the difference between c and d.

The *vinculum* ——, the *brackets* [ ], and the *brace* { } have the same signification as the parenthesis.

Thus, $\overline{a+b} \times \overline{c-d}$ is equivalent to $(a+b)(c-d)$.

The vinculum is sometimes placed in a vertical position. Thus,

$$\left.\begin{array}{r} a \\ +b \\ -c \end{array}\right| d \quad \text{is equivalent to } (a+b-c)\,d.$$

19. ***The Terms*** of an expression are the parts which are connected by the sign $+$ or the sign $-$. Thus, a, b, c, and d are the terms of the expression $a+b-c+d$.

20. ***A Polynomial*** is an expression containing two or more terms.

21. ***A Binomial*** is a polynomial containing only *two* terms. Thus, $abc+x$ is a binomial.

22. ***A Trinomial*** is a polynomial containing only *three* terms. Thus, $ab+ac-bc$ is a trinomial.

23. ***A Monomial*** is an expression which does not contain parts connected by the sign $+$ or the sign $-$. Thus, *abc* is a monomial.

24. When one quantity is the product of two or more other quantities, each of the latter is called a ***Factor*** of the product. Thus, *a*, *b*, and *c* are factors of the product *abc*.

25. ***A Numerical Factor*** is one which is expressed by a figure, or figures.

26. ***A Literal Factor*** is one which is expressed by a letter, or letters.

27. When a product contains one factor which is numerical, and another which is literal, the former factor is called the ***Coefficient*** of the latter. Thus, in the product $7abc$, 7 is the coefficient of *abc*.

When all the factors of a product are *literal*, any one of them may be considered as the coefficient of the product of the others Thus, in the product *abc*, we may consider *a* as the coefficient of *bc*, *b* as the coefficient of *ac*, and *c* as the coefficient of *ab*.

28. ***A Power*** of a quantity is the product of factors each of which is equal to that quantity. Thus, $a \times a$ is the *second power of a*; $a \times a \times a$ is the *third power of a*; $a \times a \times a \times a$ is the *fourth power of a*; and so on.

The *first power of a* is *a*.

29. ***An Exponent*** is a number placed on the right of, and a little above a quantity, and indicates how many times the quantity is to be used as a factor. Thus, a^2 is equivalent to $a \times a$; a^3 is equivalent to $a \times a \times a$; a^4 is equivalent to $a \times a \times a \times a$; and so on. If no exponent is expressed, 1 is understood. Thus, a is equivalent to a^1.

The product of n factors each equal to a is expressed by a^n, and n is called the exponent of a.

The second power of *a*, that is, a^2, is often called the *square of a*, and the third power of *a*, that is, a^3, is often called the

cube of a. The expression a^4 is read *a to the fourth power*, or briefly, *a to the fourth;* and a^n is read *a to the* n^{th}.

30. ***The Square Root*** of any given quantity is that quantity which has the *given* quantity for its *square* or *second power;* the *cube root* of any given quantity is that quantity which has the *given* quantity for its *cube* or *third power;* the *fourth root* of any given quantity is that quantity which has the *given* quantity for its *fourth power;* and so on.

31. ***The Radical Sign*** $\sqrt{}$ indicates that some root of the quantity to which it is prefixed is to be found.

32. ***The Index*** of the root is the number placed above the radical sign.

The square root of a is denoted thus, $\sqrt[2]{a}$, or simply thus, $\sqrt{a}$; the cube root of a is denoted thus, $\sqrt[3]{a}$; the fourth root of a is denoted thus, $\sqrt[4]{a}$; and so on.

33. The symbols employed in Algebra are classified as follows:

1. ***Symbols of Quantity*** are letters and other characters used to represent quantities.

2. ***Symbols of Operation*** are the signs $+$, $-$, $\times$, $\div$, $\sqrt{}$, and the exponential sign.

3. ***Symbols of Relation*** are the signs $=$, $>$, $<$, and others to be explained hereafter.

4. ***Symbols of Aggregation*** are the signs ( ), [ ], { }, —, and |.

34. ***Similar*** or ***Like Quantities*** are such as do not differ, or differ only in their numerical coefficients. Thus, $4ab$ and $10ab$ are similar.

35. ***Dissimilar*** or ***Unlike Quantities*** are such as are not similar. Thus, $4ab$ and $10a^2b$ are dissimilar.

REMARK.—An exception must be made in those cases where letters are considered as coefficients. Thus, ax^2 and bx^2 are similar if a and b are considered as coefficients.

36. Each of the literal factors of a term is called a ***Dimension*** of the term, and the ***Degree*** of a term is equal to the number of its dimensions. The degree of a term, therefore, is equal to the sum of the exponents of its literal factors. Thus, $5a^2b^3c$ is of the *sixth* degree.

37. ***A Homogeneous Polynomial*** is one whose terms are all of the same degree. Thus, $7a^3 + 3a^2b + 4abc$ is homogeneous.

38. ***The Numerical Value*** of an algebraic expression is the number obtained by substituting for each letter its numerical value, and then performing the indicated operations. Thus, if $a = 5$ and $b = 6$, the numerical value of the expression $3a - 2b$ is 3.

Some terms of frequent use in Algebra are here defined.

39. ***A Proposition*** is the statement of a truth, or of something to be done.

Propositions are of the following kinds: Axioms, theorems, lemmas, problems, postulates, corollaries, scholiums.

1. *An Axiom* is a self-evident truth.

2. *A Theorem* is a truth requiring demonstration.

3. *A Lemma* is an auxiliary theorem used in the demonstration of another theorem.

4. *A Problem* is a question proposed for solution.

5. *A Postulate* assumes the possibility of the solution of some problem.

6. *A Corollary* is an obvious consequence deduced from one or more propositions.

7. *A Scholium* is a remark upon one or more propositions.

40. ***An Hypothesis*** is a supposition, made either in the enunciation of a proposition, or in the course of a demonstration.

41. ***A Formula*** is a theorem expressed in algebraic language.

42. AXIOMS.

1. The whole is equal to the sum of all its parts.

2. If equal quantities be added to equal quantities, the sums will be equal.

3. If equal quantities be subtracted from equal quantities, the remainders will be equal.

4. If equal quantities be multiplied by the same or by equal quantities, the products will be equal.

5. If equal quantities be divided by the same or by equal quantities, the quotients will be equal.

6. Quantities that are equal to the same quantity are equal to each other.

NOTATION.

43. ***Algebraic Notation*** consists in representing quantities, operations, and relations by means of symbols.

EXAMPLES IN NOTATION.

44. Express, in algebraic language, the following eight statements:

1. The second power of a, increased by twice the product of b and c, diminished by the second power of c, and increased by d, is equal to m times x. *Ans.* $a^2 + 2bc - c^2 + d = mx$.

2. The quotient arising from dividing a by the sum of x and b, is equal to twice b diminished by c.

Ans. $\dfrac{a}{x+b} = 2b - c$.

3. One-third of the remainder obtained by subtracting four from six times x, is equal to the quotient arising from dividing five by the sum of a and b.

Ans. $\dfrac{6x-4}{3} = \dfrac{5}{a+b}$.

4. Three-fourths of the sum of x and five, is equal to three-sevenths of b, diminished by seventeen.

Ans. $\frac{3}{4}(x+5)=\frac{3}{7}b-17.$

5. One-ninth of the sum of three times x and b, added to one-third of the sum of twice x and four, is equal to the product of a, b, and c.

Ans. $\frac{1}{3}(2x+4)+\frac{1}{9}(3x+b)=abc.$

6. The quotient arising from dividing the sum of a and b by the product of c and d, is greater than p times the sum of m, n, x, and y.

Ans. $\frac{a+b}{cd}>p(m+n+x+y).$

7. The square root of the sum of a and b is equal to m times the cube root of the remainder obtained by subtracting y from x.

Ans. $\sqrt{a+b}=m\sqrt[3]{x-y}.$

8. The square root of x, diminished by the square root of y, is equal to n times the sum of the fourth root of a and the fourth root of b.

Ans. $\sqrt{x}-\sqrt{y}=n(\sqrt[4]{a}+\sqrt[4]{b}).$

45. Express, in common language, the following six algebraic expressions:

1. $$\frac{a+x}{b}+\frac{x}{c}=\frac{m}{a+b}.$$

Ans. The quotient arising from dividing the sum of a and x by b, increased by the quotient of x divided by c, is equal to the quotient of m divided by the sum of a and b.

2. $$3a^2+(b-c)(d+e)=x-y.$$

3. $$3a^2+b-c(d+e)=x-y.$$

4. $$\frac{a+x}{5+b+c}+\frac{a-y}{3}=\frac{m}{a+b}.$$

5. $$(a+b)\sqrt{b^3}-a^2c=3(a+m+x)$$

6. $$\frac{\sqrt{5b}+4\sqrt{c}}{a+2b}=5x+\frac{m}{n}.$$

46. NUMERICAL VALUES.

If $a = 1$, $b = 3$, $c = 4$, $d = 6$, $e = 2$, and $f = 0$, find the numerical value of each of the following ten expressions:

1. $a + 2b + 4c$. *Ans.* 23.

2. $3b + 5d - 2e$. *Ans.* 35.

3. $ab + 2bc + 3ed$. *Ans.* 63.

4. $ac + 4cd - 2be$. *Ans.* 88.

5. $abc + 4bd + ec - df$. *Ans.* 92.

6. $a^2 + b^2 + c^2 + f^2$. *Ans.* 26.

7. $\dfrac{cd}{b} + \dfrac{4be}{3a} - \dfrac{cd}{24}$. *Ans.* 15.

8. $c^4 - 4c^3 + 3c - 6$. *Ans.* 6.

9. $\dfrac{b^2 + c^2}{2c - 3a}$. *Ans.* 5.

10. $\sqrt{(27b)} - \sqrt[3]{(2c)} + \sqrt{(2e)}$. *Ans.* 9.

11. Find the value of $(x + y)(x - y)$, when $x = 8$ and $y = 5$. *Ans.* 39.

12. Find the value of $x + y \times x - y$, when $x = 8$ and $y = 5$. *Ans.* 43.

13. Find the value of $x\sqrt{(x^2 - 8y)} + y\sqrt{(x^2 + 8y)}$, when $x = 5$ and $y = 3$. *Ans.* 26.

14. Find the value of $(b - x)(\sqrt{a} + b) + \sqrt{\{(a-b)(x+y)\}}$, when $a = 16, b = 10$, $x = 5$, and $y = 1$. *Ans.* 76.

15. Find the value of x in the equation $x = \dfrac{(a + b)(c - d)^2}{e}$, when $a = 10$, $b = 5$, $c = 4$, $d = 2$, and $e = 3$.

Ans. $x = 20$.

47. SYNOPSIS FOR REVIEW.

CHAPTER I. DEFINITIONS AND NOTATION.

- ALGEBRA.
- ALGEBRAIC QUANTITY.
  - *Known.*
  - *Unknown.*
- SUM. REMAINDER. PRODUCT. QUOTIENT.
- EQUATION . . . .
  - *First Member.*
  - *Second Member.*
- INEQUALITY . . . .
  - *First Member.*
  - *Second Member.*
- NAMES OF EXPRESSIONS.
  - *Terms.*
  - *Polynomial* . .
    - Binomial.
    - Trinomial.
  - *Factor* . . . . .
    - Numerical.
    - Literal.
    - Coefficient.
- SYMBOLS . . . .
  - *Of quantity* . . .
    - Known: a, b, c, d, etc.
    - Unknown: t, u, v, w, x, y, z.
  - *Of operation* . .
    - $+$, $-$, $\times$, $\cdot$, $\div$, $\frac{a}{b}$, $\sqrt{\ }$, Exponent.
  - *Of relation* . . .
    - $=$, $>$, $<$.
  - *Of aggregation.*
    - $(\ )$, $\{\ \}$, $[\ ]$, $\overline{\ \ }$, $|$.
- SIMILAR QUANTITIES. DISSIMILAR QUANTITIES.
- DEGREE OF A TERM. HOMOGENEOUS POLYNOMIAL.
- NUMERICAL VALUE.
- PROPOSITION . . . . . .
  - *Axiom.*
  - *Theorem.*
  - *Problem.*
  - *Postulate.*
  - *Corollary.*
  - *Scholium.*
- HYPOTHESIS. FORMULA.

CHAPTER II.

FUNDAMENTAL PROCESSES.

ADDITION

48. ***Addition*** is the process of finding the simplest expression for the *sum* of two or more quantities.

49. ***A Positive Term*** is one which is preceded by the sign $+$. When a term has no sign prefixed, the sign $+$ is understood.

50. ***A Negative Term*** is one which is preceded by the sign $-$.

ORDER OF TERMS.

51. The value of a polynomial whose terms are positive is the same in whatever order the terms may be written. Thus, $a+b+c=a+c+b=b+c+a=c+a+b=c+b+a=b+a+c$.

52. When a polynomial contains both positive and negative terms, we may write the former terms first in any order, and the latter after them in any order. Thus, $a+b-c-e=a+b-e-c$ $=b+a-c-e=b+a-e-c$.

53. In some cases we may vary the order of terms still further. Thus, if $a=10$, $b=6$, and $c=5$, then $a+b-c=a-c+b$ $=b-c+a$.

But, if $a=2$, $b=6$, and $c=5$, the expression $a-c+b$ presents a difficulty, because we are apparently required to subtract 5 from 2. It will be convenient to agree that such an expression as $a-c+b$, when c is greater than a, shall be understood to be equivalent to $a+b-c$. At present we shall not use such an expression except when c is less than $a+b$.

In like manner we shall consider $-b+a$ as equivalent to $a-b$. We shall recur to this point in Chapter III.

REDUCTION OF SIMILAR TERMS.

54. When two or more terms of a polynomial are *similar*, it may be reduced to a simpler form.

1. Let it be required to simplify the expression $4a^2b - 3a^2c + 9ac^2 - 2a^2b + 7a^2c - 6b^2$.

This expression may be written thus: $4a^2b - 2a^2b + 7a^2c - 3a^2c + 9ac^2 - 6b^2$ (**53**). Now $4a^2b - 2a^2b = 2a^2b$, and $7a^2c - 3a^2c = 4a^2c$. Hence the given expression may be reduced to $2a^2b + 4a^2c + 9ac^2 - 6b^2$.

2. Let it be required to simplify the expression $2a^3bc^2 - 4a^3bc^2 + 6a^3bc^2 - 8a^3bc^2 + 11a^3bc^2$.

We write the positive terms in one column and the negative terms in another thus,

$$\begin{array}{l}
2a^3bc^2 - 4a^3bc^2 \\
6a^3bc^2 - 8a^3bc^2 \\
11a^3bc^2 \\
\hline
19a^3bc^2 - 12a^3bc^2 = 7a^3bc^2.
\end{array}$$

3. Let it be required to simplify the expression $4abc + 3a^2b + 2a^2b - 5a^2b - 3a^2b$.

Arranging the terms thus,

$$\begin{array}{rl}
4abc + 3a^2b - 5a^2b & \\
+ 2a^2b - 3a^2b & \text{and uniting,} \\
\hline
\text{we obtain} \quad 4abc + 5a^2b - 8a^2b &
\end{array}$$

But $4abc + 5a^2b - 8a^2b = 4abc + 5a^2b - 5a^2b - 3a^2b = 4abc - 3a^2b$.

RULE.

I. *Reduce the positive similar terms to one term by addition.*

II. *In like manner reduce the negative similar terms to one term.*

III. *Then subtract the less result from the greater, and to the remainder prefix the sign of the greater.*

EXAMPLES.

Reduce each of the following expressions to its simplest form:

1. $10a^4 + 3a^4 + 6a^4 - a^4 - 5a^4$. *Ans.* $13a^4$.

2. $5a^4b + 3\sqrt{ab^2c} - 7ab + 17ab + 2\sqrt{ab^2c} - 6a^4b - 8\sqrt{ab^2c} - 10ab + 9a^4b$. *Ans.* $8a^4b - 3\sqrt{ab^2c}$.

3. $3a - 2a - 7c + 3c + 2a + 4c - 3a$. *Ans.* 0.

4. $9b^3c - 8ac^2 + 15b^3c + 8ac + 9ac^2 - 24b^3c$. *Ans.* $ac^2 + 8ac$.

5. $6ac^2 - 5ab^3 + 7ac^2 - 3ab^3 - 13ac^2 + 18ab^3$. *Ans.* $10ab^3$.

6. $a^2b - 9ab^2 + 8a^2b + 5c - 3a^2b + 8ab^2 + 2a^2b + c + ab^2 - 8c$.

55. To find the sum of two or more quantities.

1. Let it be required to find the sum of $c - d + e$ and $a - b$.

Adding $c + e$ to $a - b$, we obtain $a - b + c + e$; we have, however, added d too much to $a - b$; hence, in order to obtain the correct sum, d must be subtracted from $a - b + c + e$. We thus obtain $a - b + c + e - d$. Therefore, to find the sum of quantities, all the terms of which are *unlike*, write them in any order, prefixing to each term its proper sign.

2. Let it be required to find the sum of

$$\begin{array}{lrrrr}
 & a^3 + 3a^2 - 4ab & & & \\
 & 2a^2 - 3ab & + b^2 & - c, & \\
\text{and} & a^2 + 2ab & - 5b^2 & + 3c. & \\
\hline
 & a^3 + 6a^2 - 5ab & - 4b^2 & + 2c &
\end{array}$$

Their sum, after reducing, is $a^3 + 6a^2 - 5ab - 4b^2 + 2c$ (**54**).

RULE.

I. *Write similar terms, with their proper signs, in the same column.*

II. *Reduce each column* (**54**), *and to the results annex those terms which cannot be reduced, prefixing to each its proper sign.*

EXAMPLES.

(1.)

$$\begin{array}{lrrr}
 & 7x & + 3ab & + 2c \\
 & -3x & - 3ab & - 5c \\
 & 5x & - 9ab & + 9c \\
\hline
\text{Sum} & 9x & - 9ab & + 6c.
\end{array}$$

(2.)

$$\begin{array}{rrr}
16a^2b^2 & + bc & - 2abc \\
-4a^2b^2 & + 9bc & + 6abc \\
-10a^2b^2 & - 12bc & + 3abc \\
\hline
2a^2b^2 & - 2bc & + 7abc.
\end{array}$$

3. Find the sum of $4a - 5b + 3c - 2d$, $a + b - 4c + 5d$, $3a - 7b + 6c + 4d$, and $a + 4b - c - 7d$. *Ans.* $9a - 7b + 4c$.

4. Find the sum of $x^3 + 2x^2 - 3x + 1$, $2x^3 - 3x^2 + 4x - 2$. $3x^3 + 4x^2 + 5$, and $4x^3 - 3x^2 - 5x + 9$. *Ans.* $10x^3 - 4x + 13$.

5. Find the sum of $x^2 - 3xy + y^2 + x + y - 1$, $2x^2 + 4xy - 3y^2 - 2x - 2y + 3$, $3x^2 - 5xy - 4y^2 + 3x + 4y - 2$, and $6x^2 + 10xy + 5y^2 + x + y$. *Ans.* $12x^2 + 6xy - y^2 + 3x + 4y$.

6. Find the sum of $x^3 - 2ax^2 + a^2x$, $x^3 + 3ax^2$, and $2x^3 - ax^2$. *Ans.* $4x^3 + a^2x$.

7. Find the sum of $4ab - x^2$, $3x^2 - 2ab$, and $2ax + 2bx$. *Ans.* $2ab + 2x^2 + 2ax + 2bx$.

8. Find the sum of $a + b + c + d$, $a + b + c - d$, $a + b - c + d$, $a - b + c + d$, and $-a + b + c + d$. *Ans.* $3a + 3b + 3c + 3d$.

9. Find the sum of $3(x^2 - y^2)$, $8(x^2 - y^2)$, $-5(x^2 - y^2) + 6(x^2 + y^2)^2$, and $7(x - y)^2$. *Ans.* $6(x^2 - y^2) + 6(x^2 + y^2)^2 + 7(x - y)^2$.

10. Find the sum of $a^m - b^n + 3x^p$, $2a^m - 3b^n - x^p$, and $a^m + 4b^n - x^q$. *Ans.* $4a^m + 2x^p - x^q$.

LAWS RESPECTING THE USE OF THE PARENTHESIS.

56. *If a polynomial has any number of its terms enclosed by a parenthesis preceded by the sign +, the parenthesis may be omitted and the value of the polynomial will not be changed.* Thus,

$$a - b + (c - d + e) = a - b + c - d + e \quad (\mathbf{55}).$$

COR.—*The value of a polynomial will not be changed by enclosing any number of its terms by a parenthesis, provided the parenthesis have the sign + prefixed.* Thus,

$$a - b + c - d + e = a - b + c + (-d + e) =$$
$$a - d + (c + e - b) = a + (-d + c + e - b).$$

SUBTRACTION.

57. ***Subtraction*** is the process of finding the simplest expression for the *difference* between two quantities.

58. ***The Minuend*** is the quantity from which the subtraction is to be made.

59. ***The Subtrahend*** is the quantity to be subtracted.

60. ***The Remainder*** is the result obtained by the subtraction.

61. To find the difference between two quantities.

1. Let it be required to subtract $5 - 3$ from 11.

$$\begin{array}{l} 11 \\ 5 - 3 \\ \hline 6 + 3 = 9. \end{array}$$

Subtracting 5 from 11 we obtain 6. This result is *too small* by 3, for the number 5 is *larger* by 3 than the number which was required to be subtracted. In order, therefore, to obtain the correct remainder, 3 must be added to 6.

2. Let it be required to subtract $c - d$ from $a + b$.

$$\begin{array}{l} a + b \\ c - d \\ \hline a + b - c + d. \end{array}$$

Subtracting c from $a + b$ we obtain $a + b - c$. This result is *too small* by d, for c is larger by d than the quantity which was required to be subtracted. In order, therefore, to obtain the correct remainder, d must be added to $a + b - c$. Hence $a + b - (c - d) = a + b - c + d$.

The same result may be obtained by *adding* $- c + d$ to $a + b$ (**55**).

RULE.

Change the sign of every term of the subtrahend from + to —, or from — to +, and add the result to the minuend.

REMARKS—1. Beginners may solve a few examples by *actually* changing the sign of every term in the subtrahend. After this, it is better merely to *conceive* such change to be made.

2. Subtraction, in Algebra, is proved in the same manner as in Arithmetic, by adding the remainder to the subtrahend; the sum should be equal to the minuend.

EXAMPLES.

(1.)

$$\begin{array}{ll} \text{From} & 3a+\ \ b \\ \text{Take} & \ \ a-\ \ b \\ \hline \text{Remainder} & 2a+2b \end{array}\Bigg\}\ \text{The same with the signs of the subtrahend changed.}\ \Bigg\{\begin{array}{r} 3a+\ \ b \\ -a+\ \ b \\ \hline 2a+2b \end{array}$$

(2.)

$$\begin{array}{ll} \text{From} & 11a^2+3ab-4xy \\ \text{Take} & 5a^2+4ab-6xy \\ \hline \text{Remainder} & 6a^2-\ ab+2xy. \end{array}$$

(3.)

$$\begin{array}{ll} \text{From} & 5a-3b+4c-7d \\ \text{Take} & 2a-2b+3c-\ d \\ \hline \text{Remainder} & 3a-\ b+\ c-6d. \end{array}$$

4. From $x^4+4x^3-2x^2+7x-1$ take $x^4+2x^3-2x^2+6x-1$. *Ans.* $2x^3+x$.
5. From $3a^2-2ax+x^2$ take a^2-ax+x^2. *Ans.* $2a^2-ax$.
6. From $2a-2b-c+d$ take $a-b-2c+2d$.
7. From $4a+3b-2c+8d$ take $a+2b+c-5d$. *Ans.* $3a+b-3c+13d$.
8. From $8x^2-3ax+5$ take $5x^2+2ax+5$.
9. From $7xy-10y+4x$ take $3xy+3y+3x$. *Ans.* $4xy-13y+x$.
10. From $3a+b+c$ take $a-b-c$. *Ans.* $2a+2b+2c$.
11. From $a^2+2ab+b^2$ take $a^2-2ab+b^2$. *Ans.* $4ab$.
12. From $a^3+3b^2c+ab^2-abc$ take $ab^2-abc+b^3$.
13. From $5x^2-y^3$ take $4x^2-y^3+y^4$. *Ans.* x^2-y^4.
14. From $4a^m+2x^p-x^q$ take $a^m-b^n+3x^p+2a^m-3b^n-x^p$. *Ans.* $a^m+4b^n-x^q$.

LAWS RESPECTING THE USE OF THE PARENTHESIS.

62. *A parenthesis preceded by the sign — may be omitted without affecting the value of the expression in which it occurs, provided the sign of every term within it be changed.* Thus,

$$a-(b-c-d+e)=a-b+c+d-e.$$

COR.—*The value of an expression will not be changed by enclosing any number of its terms by a parenthesis, preceded by the sign —, provided the sign of every term thus enclosed be changed.* Thus,

$$a-b+c+d-e=a-b+c-(-d+e)=a-(b-c-d+e)=a+c-(b-d+e).$$

EXAMPLES.

Remove the signs of aggregation from each of the following expressions:

1. $a - [b - (c - d)]$. *Ans.* $a - b + c - d$.
2. $a - [(b - c) - d]$. *Ans.* $a - b + c + d$.
3. $a + 2b - 6a - [3b - (6a - 6b)]$. *Ans.* $a - 7b$.
4. $7a - \{3a - [4a - (5a - 2a)]\}$. *Ans.* $5a$.
5. $3a - [a + b - \{a + b + c - (a + b + c + d)\}]$. *Ans.* $2a - b - d$.
6. $2x - [3y - \{4x - (5y - 6x)\}]$. *Ans.* $12x - 8y$.
7. $a - \{a - [a - (a - x)]\}$. *Ans.* x.
8. $4x - \{x - [x - (x - 3) + 3] - 3\} - \{-x - [-x - (-x + 3) + 3] - 3\}$. *Ans.* $4x + 12$.

63. SYNOPSIS FOR REVIEW.

CHAPTER II. FUNDAMENTAL PROCESSES.

- ADDITION.
  - *Positive terms.*
  - *Negative terms.*
  - *Order of terms.*
    - When preceded by +.
    - When preceded by −.
  - *Reduction of terms.*
    - When possible.
    - Rule.
  - *Sum.*
  - *Rule.*
- SUBTRACTION.
  - *Minuend.*
  - *Subtrahend.*
  - *Remainder.*
  - *Rule.*
- USE OF PARENTHESIS.
  - *To remove.*
    - When preceded by +.
    - When preceded by −.
  - *To introduce.*
    - When preceded by +.
    - When preceded by −.

MULTIPLICATION.

64. ***The Product*** of two quantities is a quantity which is as many times greater than one of them as the other is greater than a unit. Thus, the product of 5 and 4 is 20.

65. ***Multiplication*** is the process of finding the product of two quantities.

66. ***The Multiplicand*** is the quantity which is to be multiplied.

67. ***The Multiplier*** is the quantity by which to multiply.

68. The product of three or more quantities is sometimes called a *Continued Product.*

The product of any number of factors is the same in whatever *order* they may be taken; thus, $abc = acb = bca$. The literal factors are generally arranged in alphabetical order.

69. To find the Product of Monomials.

Let it be required to find the product of $7a^3b^2$ and $5a^2b^4c$.

We may indicate the multiplication thus:

$$7a^3b^2 \times 5a^2b^4c;$$

and since the product is the same in whatever order the factors are taken, we have

$$7a^3b^2 \times 5a^2b^4c = 7 \times 5 \times a^3a^2b^2b^4c = 35a^3a^2b^2b^4c.$$

Here a occurs as a factor five times, b occurs six times, and c once. Therefore the required product may be written thus:

$$35a^5b^6c. \quad \text{Hence,}$$

PRINCIPLES.—1. *The coefficient of the product of given monomials is equal to the product of their coefficients.*

2. *Every letter which occurs in any of the given factors must be written in the product with an exponent equal to the sum of all its exponents in the given factors.*

COR.—We may, if we please, indicate the product of the like powers of different letters by writing them within a parenthesis and placing the exponent over the whole. Thus,

$$a^2b^2 = (ab)^2; \text{ for } (ab)^2 = ab \times ab = aa \times bb = a^2b^2.$$

EXAMPLES.

1. Multiply ab by x. *Ans.* abx.
2. Multiply $3ax$ by $2ay$. *Ans.* $6a^2xy$.
3. Multiply $4am$ by $3bc^2n$. *Ans.* $12abc^2mn$.
4. Multiply $5a^2x$ by $3a^4x^3$. *Ans.* $15a^6x^4$.
5. Multiply $3a^mx^n$ by $9a^nx^m$. *Ans.* $27a^{m+n}x^{m+n}$

70. To find the Product of two Polynomials.

1. Let it be required to multiply $a + b$ by c. The product of a and c is ac; but this is too small by bc, for it is the *sum* of a and b which is to be multiplied by c. Hence

$$\begin{array}{l} a + b \\ c \\ \hline ac + bc. \end{array}$$

$$(a + b)\,c = ac + bc.$$

2. Let it be required to multiply $a - b$ by c.
Here the product of a and c must be *diminished* by the product of b and c. Hence

$$\begin{array}{l} a - b \\ c \\ \hline ac - bc. \end{array}$$

$$(a - b)\,c = ac - bc.$$

3. Let it be required to multiply $a + b$ by $c + d$.
The product of $a + b$ and c is $ac + bc$; but $a + b$ is to be multiplied by the sum of c and d; hence $ac + bc$ is too small by the product of $a + b$ and d; that is, by $ad + bd$, which must, therefore, be added to $ac + bc$ to produce the correct result. Hence

$$\begin{array}{l} a + b \\ c + d \\ \hline ac + bc + ad + bd. \end{array}$$

$$(a + b)\,(c + d) = ac + bc + ad + bd.$$

4. Let it be required to multiply $a + b$ by $c - d$.

Here the product of $a + b$ and c must be *diminished* by the product of $a + b$ and d. Hence

$$\begin{array}{l} a+b \\ c-d \\ \hline ac+bc-(ad+bd)=ac+bc-ad-bd. \end{array}$$

$$(a+b)(c-d) = ac + bc - (ad + bd) = ac + bc - ad - bd.$$

5. Let it be required to multiply $a - b$ by $c - d$.

Here the product of $a - b$ and c, which is $ac-bc$, is to be *diminished* by the product of $a - b$ and d, which is $ad - bd$. Hence

$$\begin{array}{l} a-b \\ c-d \\ \hline ac-bc-(ad-bd)=ac-bc-ad+bd. \end{array}$$

$$(a-b)(c-d) = ac - bc - (ad - bd) = ac - bc - ad + bd.$$

In this example, we observe that corresponding to the terms $-b$ and c, one of which occurs in the multiplicand and the other in the multiplier, there is the term $-bc$ in the product; and corresponding to $-b$ of the multiplicand and $-d$ of the multiplier, there is the term $+bd$ in the product. Hence it is often stated as an independent truth, that

$$(-b) \times c = -bc, \text{ and } (-b) \times (-d) = +bd.$$

Thus, the sign of the product is deduced from the signs of the factors by the rule,

Like signs produce +, and unlike signs produce —.

6. Let it be required to multiply $4a^2 - 5ab + 6b^2$ by $2a^2 - 3ab + 4b^2$.

$$\begin{array}{lllll} 4a^2 & -5ab & +6b^2 & & \\ 2a^2 & -3ab & +4b^2 & & \\ \hline 8a^4 & -10a^3b & +12a^2b^2 & & \\ & -12a^3b & +15a^2b^2 & -18ab^3 & \\ & & +16a^2b^2 & -20ab^3 & +24b^4 \\ \hline 8a^4 & -22a^3b & +43a^2b^2 & -38ab^3 & +24b^4. \end{array}$$

7. Let it be required to multiply $2x^2 + 3x + 4$ by $2x^2 - 3x + 4$.

$$\begin{array}{l}
2x^2 + 3x + 4 \\
2x^2 - 3x + 4 \\
\hline
4x^4 + 6x^3 + 8x^2 \\
\qquad - 6x^3 - 9x^2 - 12x \\
\qquad\qquad + 8x^2 + 12x + 16 \\
\hline
4x^4 \qquad\quad + 7x^2 \qquad\quad + 16.
\end{array}$$

RULE.

Multiply every term of the multiplicand by each term of the multiplier in succession; if a term in the multiplicand and a term in the multiplier have like signs, prefix the sign + to their product; if they have unlike signs, prefix the sign —; then take the sum of these partial products to form the complete product.

EXAMPLES.

1. Multiply $2p - q$ by $2q + p$. *Ans.* $3pq + 2p^2 - 2q^2$.

2. Multiply $a^2 + 3ab + 2b^2$ by $7a - 5b$.
Ans. $7a^3 + 16a^2b - ab^2 - 10b^3$.

3. Multiply $a^2 - ab + b^2$ by $a^2 + ab - b^2$.
Ans. $a^4 - a^2b^2 + 2ab^3 - b^4$.

4. Multiply $a^2 - ab + 2b^2$ by $a^2 + ab - 2b^2$.
Ans. $a^4 - a^2b^2 + 4ab^3 - 4b^4$.

5. Multiply $a^2 + 2ax + x^2$ by $a^2 + 2ax - x^2$.
Ans. $a^4 + 4a^3x + 4a^2x^2 - x^4$.

6. Multiply $a^2 + 4ax + 4x^2$ by $a^2 - 4ax + 4x^2$.
Ans. $a^4 - 8a^2x^2 + 16x^4$.

7. Multiply $a^2 - 2ax + bx - x^2$ by $b + x$.
Ans. $a^2b + (a - b)^2x - 2ax^2 - x^3$.

8. Multiply $15x^2 + 18ax - 14a^2$ by $4x^2 - 2ax - a^2$.
Ans. $60x^4 + 42ax^3 - 107a^2x^2 + 10a^3x + 14a^4$.

9. Multiply $2x^3 + 4x^2 + 8x + 16$ by $3x - 6$.
Ans. $6x^4 - 96$.

10. Multiply $x^5 - x^4y + xy^4 - y^5$ by $x + y$.
Ans. $x^6 - x^4y^2 + x^2y^4 - y^6$.

11. Multiply $a^2 + b^2 + c^2 + bc + ac - ab$ by $a + b - c$.
Ans. $a^3 + b^3 - c^3 + 3abc$.

12. Multiply $x^4 + 2x^3 + x^2 - 4x - 11$ by $x^2 - 2x + 3$.
Ans. $x^6 + 10x - 33$.

13. Multiply $a^4 - 2a^3 + 3a^2 - 2a + 1$ by $a^4 + 2a^3 + 3a^2 + 2a + 1$.
Ans. $a^8 + 2a^6 + 3a^4 + 2a^2 + 1$.

14. Multiply together $a - x$, $a + x$, and $a^2 + x^2$.
Ans. $a^4 - x^4$.

15. Multiply together $x - 3$, $x - 1$, $x + 1$, and $x + 3$.
Ans. $x^4 - 10x^2 + 9$.

16. Multiply together $x^2 - x + 1$, $x^2 + x + 1$, and $x^4 - x^2 + 1$.
Ans. $x^8 + x^4 + 1$.

17. Multiply together $a + x$, $b + x$, and $c + x$.
Ans. $abc + (ab + bc + ac)x + (a + b + c)x^2 + x^3$.

18. Simplify $(a + b)(b + c) - (c + d)(a + d) - (a + c)(b - d)$.
Ans. $b^2 - d^2$.

19. Simplify $(a + b + c + d)^2 + (a - b - c + d)^2 + (a - b + c - d)^2 + (a + b - c - d)^2$.
Ans. $4(a^2 + b^2 + c^2 + d^2)$.

20. Simplify $(a + b + c)^2 - a(b + c - a) - b(a + c - b) - c(a + b - c)$.
Ans. $2(a^2 + b^2 + c^2)$.

21. Prove that $x^8 + y^8 + (x + y)^8 = 2(x^2 + xy + y^2)^4 + 8x^2y^2(x + y)^2(x^2 + xy + y^2)$.

22. Prove that $4xy(x^2 + y^2) = (x^2 + xy + y^2)^2 - (x^2 - xy + y^2)^2$.

23. Multiply $(x^2 - 3x + 2)^2$ by $x^2 + 6x + 1$.
Ans. $x^6 - 22x^4 + 60x^3 - 55x^2 + 12x + 4$.

24. Multiply $(a + b)^2$ by $(a - b)^3$.
Ans. $a^5 - a^4b - 2a^3b^2 + 2a^2b^3 + ab^4 - b^5$.

25. Prove that $(a + b)^2 - (a - b)^2 = 4ab$.

71. *The square of the sum of two quantities is equal to the sum of their squares increased by twice their product.*

If we multiply $a + b$

by $a + b$

$$\begin{array}{ll} & a^2 + ab \\ & \quad\; + ab + b^2 \\ \hline \text{we obtain} & a^2 + 2ab + b^2 \end{array}$$

hence $(a+b)^2 = a^2 + 2ab + b^2$.

If we wish to obtain the square of the sum of two quantities, this theorem enables us to write the terms of the result without the necessity of performing the actual multiplication.

EXAMPLES.

1. $(2 + 5)^2 = 4 + 20 + 25 = 49$.
2. $(2m + 3n)^2 = 4m^2 + 12mn + 9n^2$.
3. $(ax + by)^2 = a^2x^2 + 2abxy + b^2y^2$.
4. $(c + 2d)^2 = c^2 + 4cd + 4d^2$.
5. $(a^2 + b^2)^2 = a^4 + 2a^2b^2 + b^4$.
6. $(a^3 + b^3)^2 = a^6 + 2a^3b^3 + b^6$.
7. $[(x + y)^m + (x - y)^n]^2 = (x + y)^{2m} + 2(x + y)^m(x - y)^n + (x - y)^{2n}$.

72. *The square of the difference between two quantities is equal to the sum of their squares diminished by twice their product.*

If we multiply $a - b$

by $a - b$

$$\begin{array}{ll} & a^2 - ab \\ & \quad\; - ab + b^2 \\ \hline \text{we obtain} & a^2 - 2ab + b^2 \end{array}$$

hence $(a-b)^2 = a^2 - 2ab + b^2$.

EXAMPLES.

1. $(5 - 3)^2 = 25 - 30 + 9 = 4$.
2. $(2x - y)^2 = 4x^2 - 4xy + y^2$.
3. $(3x - 5z)^2 = 9x^2 - 30xz + 25z^2$.

4. $(c-2d)^2 = c^2 - 4cd + 4d^2$.

5. $(a^2-b^2)^2 = a^4 - 2a^2b^2 + b^4$.

6. $[a+b-(c+d)]^2 = a^2 + 2ab + b^2 - 2(a+b)(c+d) + c^2 + 2cd + d^2$.

73. *The product of the sum and the difference of two quantities is equal to the difference between their squares.*

$$\begin{array}{ll} \text{If we multiply} & a+b \\ \text{by} & \underline{a-b} \\ & a^2+ab \\ & \underline{\quad -ab-b^2} \\ \text{we obtain} & \quad a^2-b^2; \text{ hence } (a+b)(a-b)=a^2-b^2. \end{array}$$

EXAMPLES.

1. $(3+2)(3-2) = 9 - 4 = 5$.

2. $(3a+2b)(3a-2b) = 9a^2 - 4b^2$.

3. $(m+1)(m-1) = m^2 - 1$.

4. $(c+2d)(c-2d) = c^2 - 4d^2$.

5. $(a^2+b^2)(a^2-b^2) = a^4 - b^4$.

6. $[(a+b)+c][(a+b)-c] = a^2 + 2ab + b^2 - c^2$.

7. $[(a+b)^2+(x-y)^2][(a+b)^2-(x-y)^2] = (a+b)^4 - (x-y)^4$.

74. Meaning of the Sign $\pm$.

We may here indicate the meaning of the *double* sign $\pm$, which is sometimes used.

Since $(a+b)^2 = a^2 + 2ab + b^2$, and $(a-b)^2 = a^2 - 2ab + b^2$, we may write both formulæ in the following abbreviated form:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

By this notation we are enabled to express two different theorems by *one* formula. The expression $a \pm b$ is read *a plus or minus b*.

75. By the aid of the preceding theorems the process of multiplication may often be abridged. Thus,

$$(a+b+c+d)\,(a+b-c-d)=[(a+b)+(c+d)]\,[(a+b)-(c+d)]$$
$$=(a+b)^2-(c+d)^2\ (\mathbf{73})=a^2+2ab+b^2-(c^2+2cd+d^2)\quad(\mathbf{71})$$
$$=a^2+2ab+b^2-c^2-2cd-d^2.$$

REMARKS ON MULTIPLICATION.

76. The *degree* of the product of two monomials is equal to the *sum* of the degrees of the multiplicand and multiplier, since all the factors of both monomials appear in the product (**69,** PRIN. 2). Thus, if we multiply $2a^2b$, which is of the third degree, by $3ab^3$, which is of the fourth degree, we obtain $6a^3b^4$, which is of the seventh degree. Hence, if two polynomials are homogeneous, their product will be homogeneous (**70,** 6).

77. The number of terms in the product of two polynomials, when the partial products do not contain *similar terms,* is equal to the product obtained by multiplying the number of terms in the multiplicand by the number of terms in the multiplier. Thus, if there be m terms in the multiplicand, and n terms in the multiplier, the number of terms in the product will be mn.

If the partial products contain *similar terms,* the number of terms in the product after reduction, will evidently be less than mn.

78. When the multiplicand and multiplier are arranged in the same way, according to the powers of some common letter, if there be one, the first and last terms of the product are unlike any other terms. Thus, in the sixth example of Art. **70**, the multiplicand and multiplier are arranged according to the descending powers of a; the first term of the product is $8a^4$ and the last term is $24b^4$, and there are no other terms which are *like* these; for the other terms contain a raised to some power *less* than the fourth, and thus differ from $8a^4$; and they all contain a to *some* power, and thus differ from $24b^4$. Therefore the product of two polynomials cannot contain less than two terms.

79. SYNOPSIS FOR REVIEW.

CHAPTER II—*Continued.*
MULTIPLICATION.

- TERMS USED.
  - *Multiplicand.*
  - *Multiplier.*
  - *Product.*
  - *Factors.*
- PRODUCT OF MONOMIALS.
  - *Order of factors,*
  - *Law of* . .
    - Coefficients.
    - Exponents.
- PRODUCT OF POLYNOMIALS.
  - *Investigation for rule.*
  - *Rule.*
  - *Law of Signs.*
    - Like signs.
    - Unlike signs.
- THEOREMS . .
  - $(a+b)^2$.
  - $(a-b)^2$.
  - $(a+b)(a-b)$.
- REMARKS . . .
  - *Degree of product.*
  - *Product, homogeneous when.*
  - *Number of terms in product.*
  - *First and last terms, different when.*

DIVISION.

80. ***Division*** is the converse of Multiplication. In Multiplication we determine the product of given factors. In Division we have the *product* of two factors, and one of them given to determine the other factor.

81. ***The Dividend*** is the given product.

82. ***The Divisor*** is the given factor.

83. ***The Quotient*** is the factor to be determined.

84. To find the Quotient of two Monomials.

Let it be required to divide $35a^5b^4c^2$ by $7a^3b^2c$.
The division may be indicated thus:

$$\frac{35a^5b^4c^2}{7a^3b^2c}.$$

Now, since the quotient must be such a quantity that when it is multiplied by the divisor the product shall be equal to the dividend, the coefficient of the quotient multiplied by 7 must give 35; hence, the coefficient of the quotient is found by dividing 35 by 7. Again, the exponent of any letter in the quotient added to the exponent of the same letter in the divisor, must give the exponent of this letter in the dividend (**69**, PRIN. 2); hence, the exponent of any letter in the quotient is found by subtracting its exponent in the divisor from that in the dividend. Therefore,

$$\frac{35a^5b^4c^2}{7a^3b^2c} = 5a^2b^2c. \quad \text{Hence,}$$

PRINCIPLES.—1. *The coefficient of the quotient of two given monomials is the quotient obtained by dividing the coefficient of the dividend by that of the divisor.*

2. *Every letter which occurs in the dividend must be written in the quotient, with an exponent which is found by subtracting its exponent in the divisor from that in the dividend.*

COR. $$\frac{a^m}{a^m} = a^{m-m} = a^0, \text{ and } \frac{a^m}{a^m} = 1;$$

hence, $$a^0 = 1.$$

EXAMPLES.

1. Divide abx by x. *Ans.* ab.
2. Divide $6a^2xy$ by $3ax$. *Ans.* $2ay$.
3. Divide $12abc^2mn$ by $3bc^2n$. *Ans.* $4am$.
4. Divide $15a^6x^4$ by $3a^4x^3$. *Ans.* $5a^2x$.
5. Divide $27a^{m+n}x^{m+n}$ by $9a^nx^m$. *Ans.* $3a^mx^n$.

85. It follows from Art. **84** that the exact division of monomials will be impossible:

1st. When the coefficient of the dividend is not divisible by that of the divisor.

2d. When the exponent of a letter in the divisor is greater than the exponent of the same letter in the dividend.

3d. When the divisor contains a letter that is not found in the dividend.

86. To find the Quotient of two Polynomials.

1. Let it be required to divide $ab - bc$ by b.

$$\frac{ab - bc}{b} = a - c; \text{ for } (a - c)\,b = ab - bc.$$

In this example, we observe that corresponding to the term ab in the dividend and to the divisor b, there is the term a in the quotient; and corresponding to the term $-bc$ in the dividend and to the divisor b, there is the term $-c$ in the quotient.

We have already seen that

$$b \times (-c) = -bc, \text{ and } (-b)(-c) = bc \ (\mathbf{70}).$$

In like manner, the following statements may be admitted:

$$\frac{-bc}{-c} = b, \text{ and } \frac{bc}{-c} = -b.$$

Thus the sign of the quotient is deduced from the signs of the dividend and divisor by the rule,

Like signs produce $+$*, and unlike signs produce* $-$.

2. Let it be required to divide $ab^2 - abc + abd$ by ab.

$$\frac{ab^2 - abc + abd}{ab} = b - c + d.$$

We divide each term of the dividend by the divisor, then collect the partial quotients to obtain the complete quotient.

3. Let it be required to divide $8a^4 + 43a^2b^2 - 22a^3b + 24b^4 - 38ab^3$ by $2a^2 + 4b^2 - 3ab$.

The operation may be conveniently arranged as follows:

$$
\begin{array}{rl|l}
 & \text{DIVIDEND.} & \text{DIVISOR.} \\
 & 8a^4-22a^3b+43a^2b^2-38ab^3+24b^4 & 2a^2-3ab+4b^2 \\
 & \underline{8a^4-12a^3b+16a^2b^2} & \overline{4a^2-5ab+6b^2}=\text{Quo.} \\
\text{1st Rem.} = & -10a^3b+27a^2b^2-38ab^3+24b^4 & \\
 & \underline{-10a^3b+15a^2b^2-20ab^3} & \\
\text{2d Rem.} = & 12a^2b^2-18ab^3+24b^4 & \\
 & \underline{12a^2b^2-18ab^3+24b^4} &
\end{array}
$$

Now, the term of the dividend, which contains the highest power of any letter as a, must be equal to the product arising from multiplying the term of the divisor which contains the highest power of that letter by the term of the quotient which contains the highest power of the same letter. Therefore, if we arrange the dividend and divisor according to the descending powers of a common letter as a, the first term of the quotient is found by *dividing* the first term of the dividend by the first term of the divisor. Hence, in this example the first term of the quotient is $\dfrac{8a^4}{2a^2} = 4a^2$.

Again, the dividend is equal to the sum of the partial products obtained by multiplying the divisor by each term of the quotient in succession; and, therefore, if the product of the divisor by the term just found is subtracted from the dividend, the remainder must be equal to the sum of the partial products obtained by multiplying the divisor by the remaining terms of the quotient, and hence may be used as a new dividend to obtain the second term of the quotient. Proceeding in this manner, we find the complete quotient to be

$$4a^2 - 5ab + 6b^2.$$

A similar course of reasoning is applicable when the dividend and divisor are arranged according to the *ascending* powers of a common letter.

RULE.

I. *Arrange the dividend and divisor according to the powers of some common letter.*

II. *Divide the first term of the dividend by the first term of the divisor; the result will be the first term of the quotient. Multiply the whole divisor by this term, and subtract the product from the dividend.*

III. *Regard the remainder as a new dividend; find the second term of the quotient in the same manner, and proceed with it as with the first term; and so on.*

REMARKS.—1. The situation of the divisor in regard to the dividend is a matter of arbitrary arrangement; but by placing it on the right it is more easily multiplied by the several terms of the quotient as they are found.

2. When there are more than two terms in the quotient, it is not necessary to bring down any more terms of the remainder, at each successive subtraction, than are required by the quantity to be subtracted.

3. It is evident that the exact division of one polynomial by another will be impossible, when the first term of the arranged dividend is not exactly divisible by the first term of the arranged divisor; when the last term of arranged dividend is not divisible by the last term of the arranged divisor, or when the first term of any arranged remainder is not divisible by the first term of the divisor.

EXAMPLES.

1. Divide $x^3 + 1$ by $x + 1$. *Ans.* $x^2 - x + 1$.

2. Divide $27x^3 + 8y^3$ by $3x + 2y$. *Ans.* $9x^2 - 6xy + 4y^2$.

3. Divide $a^3 - 2ab^2 + b^3$ by $a - b$. *Ans.* $a^2 + ab - b^2$.

4. Divide $a^3 - 2a^2b - 3ab^2$ by $a + b$. *Ans.* $a^2 - 3ab$.

5. Divide $64x^6 - y^6$ by $2x - y$.
Ans. $32x^5 + 16x^4y + 8x^3y^2 + 4x^2y^6 + 2xy^4 + y^5$.

6. Divide $a^5 + b^5$ by $a + b$.
Ans. $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.

7. Divide $a^6 - 16a^3x^3 + 64x^6$ by $4x^2 + a^2 - 4ax$.
Ans. $16x^4 + 16ax^3 + 12a^2x^2 + 4a^3x + a^4$.

8. Divide $1 - 18z^2 + 81z^4$ by $1 + 6z + 9z^2$.
Ans. $1 - 6z + 9z^2$.

9. Divide $81a^8 + 16b^{12} - 72a^4b^6$ by $9a^4 + 12a^2b^3 + 4b^6$.
Ans. $9a^4 - 12a^2b^3 + 4b^6$.

10. Divide $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$ by $x^3 - y^3$.
Ans. $x^2 - xy + y^2$.

11. Divide $x^4 + x^3 - 4x^2 + 5x - 3$ by $x^2 + 2x - 3$.
Ans. $x^2 - x + 1$.

12. Divide $a^4 + 2a^2b^2 + 9b^4$ by $a^2 + 2ab + 3b^2$.
Ans. $a^2 - 2ab + 3b^2$.

13. Divide $a^6 - b^6$ by $a^3 + 2a^2b + 2ab^2 + b^3$.
Ans. $a^3 - 2a^2b + 2ab^2 - b^3$.

14. Divide $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.
Ans. $x^4 + 2x^3 + 3x^2 + 2x + 1$.

15. Divide $a^3 + a^2b + a^2c - abc - b^2c - bc^2$ by $a^2 - bc$.
Ans. $a + b + c$.

16. Divide $a^3 + b^3 - c^3 + 3abc$ by $a + b - c$.
Ans. $a^2 + b^2 + c^2 + ac + bc - ab$.

17. Divide $1 - 9x^8 - 8x^9$ by $1 + 2x + x^2$.
Ans. $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7$

18. Divide $(a + b - c)(a - b + c)(b + c - a)$ by $a^2 - b^2 - c^2 + 2bc$.
Ans. $b + c - a$.

19. Divide $(a^2 - bc)^3 + 8b^3c^3$ by $a^2 + bc$.
Ans. $a^4 - 4a^2bc + 7b^2c^2$.

20. Divide the product of $x^3 - 2x + 1$ and $x^3 - 3x + 2$ by $x^3 - 3x^2 + 3x - 1$.
Ans. $x^3 + 3x^2 + x - 2$.

21. Divide the product of $a^2 + ax + x^2$ and $a^3 + x^3$ by $a^4 + a^2x^2 + x^4$.
Ans. $a + x$.

22. Divide $a^2(b + c) - b^2(a + c) + c^2(a + b) + abc$ by $a - b + c$.
Ans. $ab + bc + ac$.

23. Divide $ax^2 - ab^2 + b^2x - x^3$ by $(x + b)(a - x)$.
Ans. $x - b$.

24. Show that $(x^2 - xy + y^2)^3 + (x^2 + xy + y^2)^3$ is divisible by $2x^2 + 2y^2$.

87. Divisibility of Quantities of the form of $x^n \pm a^n$.

1. $x^n - a^n$ is divisible by $x - a$ when n is a positive integer.

2. $x^n - a^n$ is divisible by $x + a$ when n is an *even* positive integer.

3. $x^n + a^n$ is divisible by $x + a$ when n is an *odd* positive integer.*

EXAMPLES.

1. Divide $x^2 - a^2$ by $x + a$. *Ans.* $x - a$.
2. Divide $x^3 + a^3$ by $x + a$. *Ans.* $x^2 - ax + a^2$.
3. Divide $x^4 - a^4$ by $x + a$. *Ans.* $x^3 - ax^2 + a^2x - a^3$.
4. Divide $x^5 + a^5$ by $x + a$.
   Ans. $x^4 - ax^3 + a^2x^2 - a^3x + a^4$.
5. Divide $a^5 - b^5$ by $a - b$.
   Ans. $a^4 + a^3b + a^2b^2 + ab^3 + b^4$.
6. Divide $a^6 - b^5$ by $a + b$.
   Ans. $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5$.
7. Divide $x^3 + 1$ by $x + 1$. *Ans.* $x^2 - x + 1$.
8. Divide $x^5 + 1$ by $x + 1$. *Ans.* $x^4 - x^3 + x^2 - x + 1$.
9. Divide $x^2 - 1$ by $x + 1$. *Ans.* $x - 1$.
10. Divide $x^4 - 1$ by $x + 1$. *Ans.* $x^3 - x^2 + x - 1$.
11. Divide $x^6 - 1$ by $x + 1$.
    Ans. $x^5 - x^4 + x^3 - x^2 + x - 1$.
12. Divide $x^2 - 1$ by $x - 1$. *Ans.* $x + 1$.
13. Divide $x^3 - 1$ by $x - 1$. *Ans.* $x^2 + x + 1$.
14. Divide $x^4 - 1$ by $x - 1$. *Ans.* $x^3 + x^2 + x + 1$.
15. Divide $x^5 - 1$ by $x - 1$. *Ans.* $x^4 + x^3 + x^2 + x + 1$.

The student should carefully observe the law of the signs and exponents in the preceding examples.

* In Chapter XVII we shall give a *general* proof of these statements. It will be easy for the student to verify them in any particular case.

FACTORING.

88. ***Factoring*** is the process of resolving a quantity into its factors.

89. ***A Prime Quantity*** is one which is exactly divisible only by itself and by 1. Thus, x, y, and $a + b$ are prime quantities; but xy and $ax + az$ are not prime.

90. Two quantities are said to be *prime to each other,* or *relatively prime,* when they have no common factor. Thus, ab and cd are relatively prime.

The unit 1 is not generally considered as a factor.

91. ***A Composite Quantity*** is one which is the product of two or more factors. Thus, $a^2 - b^2$ is a composite quantity, the factors of which are $a + b$ and $a - b$.

92. To resolve a monomial into its prime factors.

RULE.

To the prime factors of the numerical coefficient annex the prime factors of the literal part.

EXAMPLES.

1. Resolve $12a^2b$ into its prime factors. *Ans.* $2 \times 2 \times 3aab$.

2. Resolve $18ab^2$ into its prime factors. *Ans.* $2 \times 3 \times 3abb$.

3. Resolve $21m^3n^2x$ into its prime factors. *Ans.* $7 \times 3mmmnnx$.

4. Resolve $49a^2bx^2y^3$ into its prime factors. *Ans.* $7 \times 7aabxxyyy$.

5. Resolve $210ax^3yz^2$ into its prime factors. *Ans.* $2 \times 3 \times 5 \times 7axxxyzz$.

6. Resolve $26m^4x^2yz$ into its prime factors. *Ans.* $2 \times 13mmmmxxyz$.

93. **To resolve a polynomial into two factors, one of which shall be a monomial.**

RULE.

Divide the given quantity by any monomial that will exactly divide each of its terms; the divisor will be one factor, and the quotient the other.

EXAMPLES.

Resolve each of the following expressions into two factors, one of which shall be a monomial:

1. $a + ax$. *Ans.* $a\,(1 + x)$.
2. $xz + yz$. *Ans.* $z\,(x + y)$.
3. $x^2y + xy^2$. *Ans.* $xy\,(x + y)$.
4. $6ab^2 + 9a^2bc$. *Ans.* $3ab\,(2b + 3ac)$.
5. $25a^4 - 30a^3b + 15a^2b^2$. *Ans.* $5a^2\,(5a^2 - 6ab + 3b^2)$.
6. $24a^2b^2cx - 30a^8b^5c^6y + 36a^7b^8cd + 6abc$.
   Ans. $6abc\,(4abx - 5a^7b^4c^5y + 6a^6b^7d + 1)$.
7. $3a^2x + 6abx + 3b^2x$. *Ans.* $3x\,(a^2 + 2ab + b^2)$.
8. $5 - 5y$. *Ans.* $5\,(1 - y)$.
9. $42a^2b^2 - 7abcd + 7abd$. *Ans.* $7ab\,(6ab - cd + d)$.
10. $ab^2c + 5ab^3 + ab^2c^2$. *Ans.* $ab^2\,(c + 5b + c^2)$.
11. $cx - 3cxz + cx^2$. *Ans.* $cx\,(1 - 3z + x)$.
12. $12c^4bx^3 - 15c^3x^4 - 6c^2x^3y$. *Ans.* $3c^2x^3\,(4c^2b - 5cx - 2y)$.

In resolving a polynomial into two factors, one of which shall be a monomial, it is common to divide by the *greatest* monomial that will exactly divide each of its terms; but it is not *necessary* to do this. Thus, $x^2y + xy^2$ may be expressed under any one of the three following forms:

$$xy\,(x + y), \qquad x\,(xy + y^2), \qquad y\,(x^2 + xy).$$

PRINCIPLES USED IN FACTORING BINOMIALS.

94. *The difference between the squares of two quantities is equal to the product of the sum and the difference of the quantities* (**73**). Thus,

$$a^2 - b^2 = (a + b)(a - b).$$

95. *The difference between any two like powers of two quantities is divisible by the difference between the quantities* (**87**). Thus,

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2; \text{ whence, } a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

96. *The difference between any two like even powers of two quantities is divisible by the sum of the quantities* (**87**). Thus,

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2b + ab^2 - b^3; \text{ whence, } a^4 - b^4 =$$
$$(a + b)(a^3 - a^2b + ab^2 - b^3).$$

97. *The sum of any two like odd powers of two quantities is divisible by the sum of the quantities* (**87**). Thus,

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2; \text{ whence, } a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

EXAMPLES.

98. Resolve each of the following expressions into its prime factors:

1. $a^2 - c^2$. *Ans.* $(a + c)(a - c)$.
2. $4x^2 - y^2$. *Ans.* $(2x + y)(2x - y)$.
3. $z^3 + 1$. *Ans.* $(z + 1)(z^2 - z + 1)$.
4. $a^4 - b^4$. *Ans.* $(a^2 + b^2)(a + b)(a - b)$.
5. $x^5 + y^5$. *Ans.* $(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.
6. $a^8 - c^8$. *Ans.* $(a^4 + c^4)(a^2 + c^2)(a + c)(a - c)$.
7. $x^8 - y^4$. *Ans.* $(x^4 + y^2)(x^2 + y)(x^2 - y)$.
8. $1 - c^4$. *Ans.* $(1 + c^2)(1 + c)(1 - c)$.
9. $27x^3 + 1$. *Ans.* $(3x + 1)(9x^2 - 3x + 1)$.
10. $8x^3 - 1$. *Ans.* $(2x - 1)(4x^2 + 2x + 1)$.

99. Certain trinomials can be factored in accordance with the following principle:

If two terms of a trinomial are positive squares, and the other term is twice the product of the square roots of these two, the trinomial is equal to the square of the SUM, *or the square of the* DIFFERENCE, *of these square roots, according as that other term is positive or negative* (**71-72**). Thus,

$$a^2 \pm 2ab + b^2 = (a \pm b)^2 = (a \pm b)(a \pm b).$$

EXAMPLES.

Resolve each of the following ten expressions into two equal factors:

1. $x^2 + 2ax + a^2$. *Ans.* $(x + a)(x + a)$.
2. $m^4 + n^4 + 2m^2n^2$. *Ans.* $(m^2 + n^2)(m^2 + n^2)$.
3. $16a^6b^4m^2 - 8a^3b^2m + 1$. *Ans.* $(4a^3b^2m - 1)(4a^3b^2m - 1)$.
4. $36a^2 + 12ab + b^2$. *Ans.* $(6a + b)(6a + b)$.
5. $c^2 - 10cd + 25d^2$. *Ans.* $(c - 5d)(c - 5d)$.
6. $a^2x^4 + 2ax^2y + y^2$. *Ans.* $(ax^2 + y)(ax^2 + y)$.
7. $25x^2y^4 + 20xy^2z + 4z^2$. *Ans.* $(5xy^2 + 2z)(5xy^2 + 2z)$.
8. $9x^4 - 6x^2z^2 + z^4$. *Ans.* $(3x^2 - z^2)(3x^2 - z^2)$.
9. $(a + b)^2 - 2(a + b)(c + d) + (c + d)^2$.
   Ans. $[a + b - (c + d)][a + b - (c + d)]$.
10. $a^{2m} + 2a^m b^n + b^{2n}$. *Ans.* $(a^m + b^n)(a^m + b^n)$.
11. Can $x^2 - 2xy - y^2$ be resolved into two equal factors?
12. Resolve $4b^2c^2 - (b^2 + c^2 - a^2)^2$ into its prime factors.

Here we have the difference between two squares; hence,

$$4b^2c^2 - (b^2 + c^2 - a^2)^2 = (2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2).$$

But,

$$2bc + b^2 + c^2 - a^2 = (b + c)^2 - a^2 = (b + c + a)(b + c - a), \text{ and}$$

$$2bc - b^2 - c^2 + a^2 = a^2 - (b^2 - 2bc + c^2) = a^2 - (b - c)^2 = (a + b - c)(a - b + c).$$

Therefore,

$$4b^2c^2 - (b^2 + c^2 - a^2)^2 = (b + c + a)(b + c - a)(a + b - c)(a - b + c).$$

13. Resolve $m^2 + 2mn + n^2 - a^2 + 2ab - b^2$ into its prime factors.

This expression may be put under the form,

$$m^2 + 2mn + n^2 - (a^2 - 2ab + b^2).$$

But

$$m^2 + 2mn + n^2 = (m + n)^2, \quad \text{and} \quad a^2 - 2ab + b^2 = (a - b)^2;$$

hence,

$$m^2 + 2mn + n^2 - a^2 + 2ab - b^2 = (m + n)^2 - (a - b)^2 =$$
$$(m + n + a - b)(m + n - a + b).$$

100. The following formulæ may be verified by performing the operations indicated in their second members:

$$x^2 + (a + b)x + ab = (x + a)(x + b) \quad . \quad . \quad . \quad (1),$$
$$x^2 - (a + b)x + ab = (x - a)(x - b) \quad . \quad . \quad . \quad (2),$$
$$x^2 + (a - b)x - ab = (x + a)(x - b) \quad . \quad . \quad . \quad (3),$$
$$x^2 - (a - b)x - ab = (x - a)(x + b) \quad . \quad . \quad . \quad (4).$$

From (1) and (2) it follows that

Any trinomial of the form of $x^2 + mx + n$, *or of the form of* $x^2 - mx + n$, *can be resolved into two binomial factors, if the coefficient of the second term is equal to the sum of two quantities whose product is equal to the third term.*

From (3) and (4) it follows that

Any trinomial of the form of $x^2 + mx - n$, *or of the form of* $x^2 - mx - n$, *can be resolved into two binomial factors, if the coefficient of the second term is equal to the difference of two quantities whose product is equal to the third term.*

It will be observed that we have used the words *sum* and *difference* in their arithmetical sense.

In the first form both of the terms in each binomial factor are positive.

In the second form the *second term* of each of the binomial factors is negative.

In the third form the second terms of the binomial factors have contrary signs, the larger being positive.

In the fourth form the second terms of the binomial factors have contrary signs, the larger being negative.

EXAMPLES.

1. Resolve $x^2 + 5x + 6$ into two binomial factors.

This comes under the first form. Let us now seek two numbers whose sum is 5 and product 6. We see that these numbers are 2 and 3; hence,

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

2. Resolve $x^2 - 9x + 20$ into two binomial factors.

This comes under the second form; and, therefore, since $4 + 5 = 9$, and $4 \times 5 = 20$, we have

$$x^2 - 9x + 20 = (x - 4)(x - 5).$$

3. Resolve $x^2 + 4x - 32$ into two binomial factors.

This comes under the third form; and, therefore, since $8 - 4 = 4$, and $8 \times 4 = 32$, we have

$$x^2 + 4x - 32 = (x + 8)(x - 4).$$

4. Resolve $x^2 - 5x - 66$ into two binomial factors.

This comes under the fourth form; and, therefore, since $11 - 6 = 5$, and $11 \times 6 = 66$, we have

$$x^2 - 5x - 66 = (x + 6)(x - 11).$$

Resolve each of the following ten expressions into two binomial factors:

5. $x^2 + 8x + 15$. *Ans.* $(x + 3)(x + 5)$.
6. $x^2 + 8x + 7$. *Ans.* $(x + 1)(x + 7)$.
7. $x^2 - x - 6$. *Ans.* $(x + 2)(x - 3)$.
8. $x^2 + 3x + 2$. *Ans.* $(x + 2)(x + 1)$.
9. $x^2 - x - 72$. *Ans.* $(x + 8)(x - 9)$.
10. $x^2 - 13x + 42$. *Ans.* $(x - 7)(x - 6)$.
11. $x^2 - x - 42$. *Ans.* $(x - 7)(x + 6)$.

12. $x^2 - x - 2$. *Ans.* $(x+1)(x-2)$.
13. $x^2 + 2x - 35$. *Ans.* $(x-5)(x+7)$.
14. $x^2 - x - 30$. *Ans.* $(x+5)(x-6)$.

101. Since $x^{2p}+(a+b)x^p+ab=(x^p+a)(x^p+b)$ . . . (1),

$x^{2p}-(a+b)x^p+ab=(x^p-a)(x^p-b)$ . . . (2),

$x^{2p}+(a-b)x^p-ab=(x^p+a)(x^p-b)$ . . . (3),

and $x^{2p}-(a-b)x^p-ab=(x^p-a)(x^p+b)$ . . . (4),

it follows that such expressions as x^4+8x^2+15, x^6-13x^3+42, x^8+3x^4+2, and $x^{12}-5x^6-66$ may be resolved into binomial factors in the same manner as the examples of the preceding Article. Thus,

$x^4+8x^2+15=(x^2+3)(x^2+5)$, $x^6-13x^3+42=(x^3-7)(x^3-6)$,
$x^8+3x^4+2=(x^4+2)(x^4+1)$, $x^{12}-5x^6-66=(x^6+6)(x^6-11)$.

MISCELLANEOUS EXAMPLES.

102. Resolve each of the following expressions into its prime factors:

1. $x^3 - x$. *Ans.* $(x-1)(x+1)x$.
2. $3ax^2 + 6axy + 3ay^2$. *Ans.* $3a(x+y)(x+y)$.
3. $2cx^2 - 12cx + 18c$. *Ans.* $2c(x-3)(x-3)$.
4. $27a - 18ax + 3ax^2$. *Ans.* $3a(3-x)(3-x)$.
5. $3m^3n - 3mn^3$. *Ans.* $3mn(m+n)(m-n)$.
6. $2x^2 + 6x - 8$. *Ans.* $2(x+4)(x-1)$.
7. $2x^3 + 4x^2 - 70x$. *Ans.* $2x(x+7)(x-5)$.
8. $a^2 - b^2 - c^2 - 2bc$. *Ans.* $(a+b+c)(a-b-c)$.
9. $ac + ad + bd + bc$.
   Ans. $a(c+d) + b(c+d) = (a+b)(c+d)$.
10. $am + 2bx + 2ax + bm$.
    Ans. $a(m+2x) + b(m+2x) = (a+b)(m+2x)$.
11. $a^3 - ab^2$. *Ans.* $a(a+b)(a-b)$.
12. $7x^2 - 12x + 5$.
    Ans. $x(7x-5) - (7x-5) = (x-1)(7x-5)$.
13. $x^3 - x^2 - 2x$. *Ans.* $x(x+1)(x-2)$.
14. $x^4 - 10x^2 + 9$.
    Ans. $(x^2-9)(x^2-1) = (x+3)(x-3)(x+1)(x-1)$.
15. $x^4 - 17x^2 + 16$. *Ans.* $(x+4)(x-4)(x+1)(x-1)$.

103. SYNOPSIS FOR REVIEW.

CHAPTER II.—*Continued.*

- DIVISION.
  - RELATION TO MULTIPLICATION.
  - TERMS USED
    - DIVIDEND.
    - DIVISOR.
    - QUOTIENT.
  - MONOM. ÷ MONOM.
    - LAW OF COEFFICIENTS.
    - LAW OF EXPONENTS.
    - LAW OF SIGNS
      - *Like.*
      - *Unlike.*
    - WHEN IMPOSSIBLE.
      - *1. Coefficient.*
      - *2. Exponent.*
      - *3. Literal part.*
  - POLYNOM. ÷ POLYNOM.
    - INVESTIGATION FOR RULE.
    - RULE.
    - PROOF.
    - WHEN IMPOSSIBLE.
- FACTORING.
  - TERMS USED
    - PRIME QUANTITY.
    - RELATIVELY PRIME.
    - COMPOSITE QUANTITY.
    - PRIME FACTOR.
  - MONOMIALS—RULE.
  - POLYNOMIAL WITH MONOMIAL FACTOR—RULE.
  - BINOMIALS—PRINCIPLES.
  - TRINOMIALS
    - FIRST FORM.
    - SECOND FORM.
    - THIRD FORM.
    - FOURTH FORM.

CHAPTER III.

POSITIVE AND NEGATIVE QUANTITIES.

104. In Algebra we are sometimes led to a subtraction which cannot be performed, because the subtrahend is greater than the minuend. In the equation

$$a - (b + c) = a - b - c,$$

it is *implied* that $b + c$ is less than a; but suppose that $a = 7$, $b = 7$, and $c = 3$; we shall then have

$$7 - 10 = 7 - 7 - 3 = -3.$$

In writing this equation, we may be understood to make the following statement: *It is impossible to take* 10 *from* 7*; but if* 7 *be taken from* 10, *the remainder will be* 3.

105. It might at first seem unlikely that such an expression as $7 - 10$ should occur in practice; or that if it did occur, it would only arise either from a mistake which could be instantly corrected, or from an operation being proposed which it was obviously impossible to perform, and which must therefore be abandoned. As we proceed, we shall find, however, that such expressions occur frequently. It might happen that $a - b$ appeared at the beginning of a long investigation, and that it was not easy to decide, at once, whether a were greater or less than b. The object of this chapter is to show that in such a case we may proceed on the hypothesis that a is greater than b, and that if it should finally appear that a is less than b, we shall still be able to make use of our investigation.

106. Suppose a merchant to gain in one year a certain number of dollars, and to lose a certain number of dollars in the following year; what change has taken place in his capital?

Let a denote the number of dollars gained in the first year, and b the number of dollars lost in the second year. Then if a is greater than b, the capital has been *increased* by $a - b$ dollars. If b is greater than a, the capital has been *diminished* by $b - a$ dollars. In this latter case $a - b$ is the indication of what would be pronounced, in Arithmetic, to be an impossible subtraction; but, in Algebra, it is found convenient to indicate the change in the capital by $a - b$, whether a is greater or less than b, which we may do by means of an appropriate system of *interpretation.* Thus, if $a = \$400$ and $b = \$500$, the merchant's capital has suffered a diminution of \$100. The algebraist indicates this in symbols thus:

$$400 - 500 = -100;$$

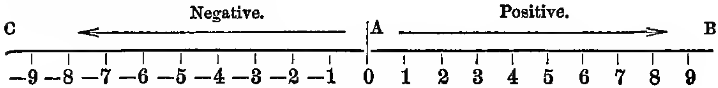
and he may convert his symbols into words by saying that the capital has been increased by $-$ \$100. This language is far removed from that of ordinary life; but if the algebraist understands it and uses it consistently, his deductions will be sound.

107. There are numerous instances in which it is convenient to be able to represent, not only the *magnitude,* but also what may be called the *quality* of the things about which we may be reasoning. In business transactions a sum of money may be *gained* or it may be *lost;* in a question of chronology we may have to distinguish a date *before* a given epoch from a date *after* that epoch; in a question of position we may have to distinguish a distance measured to the *north* of a certain point from a distance measured to the *south* of it; and so on. These pairs of related magnitudes the algebraist distinguishes by means of the signs $+$ and $-$. Thus, if the things to be distinguished are *gain* and *loss,* he may denote by $+a$ a gain of a dollars, and then he will denote by $-a$ a loss of the same extent.

108. In Arithmetic we consider only the numbers represented by the symbols 1, 2, 3, 4, etc., and intermediate fractions.

In Algebra, besides these, we consider another set of symbols, — 1, — 2, — 3, — 4, etc., and intermediate fractions.

The relation between positive and negative quantities is exhibited to the eye in the following diagram, where the distance from the zero point A to any point in the indefinite line BC is considered positive or negative according as that point is on the right or on the left of A:



109. In the preceding chapter we have given rules for the Addition, Subtraction, Multiplication, and Division of algebraic expressions. Those rules were based on arithmetical notions, and were shown to be true so long as the expressions represented *positive quantities.* Thus, when we introduced such an expression as $a - b$, we supposed a and b to be positive quantities, and a to be greater than b. But as we wish hereafter to include negative quantities among the subjects of our reasoning, it becomes necessary to recur to the consideration of these primary operations. Now it is found convenient to have the laws of the fundamental operations the same whether the symbols denote *positive* or *negative* quantities, and we may secure this convenience by suitable definitions.

110. ***The Absolute Value*** of a quantity is the number represented by that quantity taken independently of the sign which precedes it. Two quantities are *equal* when they have the same absolute value and are preceded by like signs. Two quantities may have the same absolute value and be unequal. Thus, $+7$ and -7 have the same absolute value, but they are not equal. Such quantities as $+7$ and -7 are sometimes said to be *numerically equal.*

111. In Arithmetic the object of addition is to find a number which shall contain as many units as all the given numbers taken together. This notion is not applicable to negative quantities; that is, we have as yet no *meaning* for the phrase "add -3 to $+5$," or "add -3 to -5." We shall therefore give a meaning

to the word *add* in such cases, and the meaning we propose is determined by the following

RULES.

I. *To add two quantities with like signs, add their absolute values, and prefix the common sign to the sum.*

II. *To add two quantities with unlike signs, subtract the less absolute value from the greater, and prefix to the remainder the sign of that quantity which has the greater absolute value.*

Thus, the sum of 3 and 5 is 8; the sum of -3 and -5 is -8; the sum of -3 and 5 is 2; and the sum of 3 and -5 is -2.

112. That the rules of the preceding Article are not altogether arbitrary will appear from the following illustrations:

1. Suppose a man starts from A in the line BC (**108**), and travels first 3 miles toward the right, and then 5 miles further in the same direction; his final distance from A will be 8 miles in the positive direction. This may be considered as an interpretation of the 8 obtained by adding 3 to 5.

2. Suppose a man starts from A and travels first 3 miles toward the left, and then 5 miles further in the same direction; his final distance from A will be 8 miles in the negative direction. This may be considered as an interpretation of the -8 obtained by adding -3 to -5.

3. Suppose a man starts from A and travels first 3 miles toward the left, and then turns and travels 5 miles toward the right; his final distance from A will be 2 miles in the positive direction. This may be considered as an interpretation of the 2 obtained by adding -3 to 5.

4. Suppose a man starts from A and travels first 3 miles toward the right, and then turns and travels 5 miles toward the left; his final distance from A will be 2 miles in the negative direction. This may be considered as an interpretation of the -2 obtained by adding 3 to -5.

113. In Algebra, *addition* does not necessarily imply augmentation in an arithmetical sense; nevertheless, the word *sum* is used to denote the result. Sometimes, when there might be an uncertainty on the point, the phrase *algebraic sum* is used to distinguish such a result from the *arithmetical sum* which would be obtained by the addition of the *absolute values* of the terms considered.

114. In arithmetical subtraction we have to take one number, which is called the *subtrahend*, from another, which is called the *minuend*, and the result is called the remainder. The remainder, then, may be defined as that number which must be added to the subtrahend to produce the minuend, and the object of subtraction is to find this *remainder*.

We shall use the same definition in algebraic subtraction; that is, we say that in subtraction, we have to find the quantity which must be added to the subtrahend to produce the minuend.

RULE.

Change the sign of every term in the subtrahend, and add the result to the minuend; the sum thus obtained will be the remainder required.

115. By the rule of Art. **114**, the following results are obtained:

1. Subtracting 3 from 8, we obtain 5.
2. Subtracting 8 from 3, we obtain -5.
3. Subtracting -3 from -8, we obtain -5.
4. Subtracting -3 from 8, we obtain 11.
5. Subtracting 8 from -3, we obtain -11.

Let us now recur to the diagram (**108**) and see how these results are to be interpreted.

1. Starting from the subtrahend 3, we must move a distance of 5 toward the *right*—that is, in the *positive direction*—in order to reach the minuend 8; hence, the remainder is 5.

2. Starting from the subtrahend 8, we must move a distance of 5 toward the *left*—that is, in the *negative direction*—in order to reach the minuend 3; hence, the remainder is -5.

3. Starting from the subtrahend -3, we must move a distance of 5 toward the *left*, in order to reach the minuend -8; hence, the remainder is -5.

4. Starting from the subtrahend -3, we must move a distance of 11 toward the *right*, in order to reach the minuend 8; hence, the remainder is 11.

5. Starting from the subtrahend 8, we must move a distance of 11 toward the *left*, in order to reach the minuend -3; hence, the remainder is -11.

116. In the multiplication of one monomial by another there are four cases to be considered.

1st. When the multiplicand and multiplier are positive.

2d. When the multiplicand is negative and the multiplier positive.

3d. When the multiplicand is positive and the multiplier negative.

4th. When the multiplicand and multiplier are negative.

It was shown in Art. **70** that

$$(a-b)(c-d) = ac - ad - bc + bd \quad . \quad . \quad . \quad (1)$$

Now, although the result was obtained on the supposition that $a > b$ and $c > d$, it will be convenient to assume that (1) is true *for all values of the letters*. In this way uniformity of results will be secured.

Suppose $b = 0$, and $d = 0$; then (1) becomes

$$(a-0)(c-0) = ac - a \times 0 - 0 \times c + 0 \times 0;$$

that is, $$a \times c = ac.$$

Suppose $a = 0$, and $d = 0$; then (1) becomes

$$(-b)(+c) = -bc.$$

Suppose $b = 0$, and $c = 0$; then (1) becomes

$$a(-d) = -ad.$$

Suppose $a = 0$, and $c = 0$; then (1) becomes

$$(-b)(-d) = bd.$$

Hence, to multiply one monomial by another, we have the following

RULE.

Multiply without considering the signs, and prefix $+$ *or* $-$ *to the product, according as the two monomials have like signs or unlike signs.*

117. In division we have the product of two factors, and one of them given to find the other. Therefore, since the product of the divisor and quotient must be equal to the dividend, we have for the sign of the quotient the following

RULE.

When the dividend and divisor have like signs, the quotient must have the sign $+$ *; when the dividend and divisor have unlike signs, the quotient must have the sign* $-$.

118. The words *greater* and *less* are often used in Algebra in an extended sense. We consider a greater than b, or b less than a, when $a - b$ is a *positive quantity.* This is consistent with ordinary language when a and b are positive numbers, and it is found convenient to extend the meaning of the words *greater* and *less*, so that we may still consider a greater than b, when a or b is negative, or when both are negative. Thus, in *algebraic language*, 1 is greater than -2, and -2 is greater than -3; for $1-(-2) = +3$, and $-2-(-3) = +1$ (**114**).

In this *extended or algebraic sense* a negative quantity may be said to be *less* than zero. Thus, -2 is *algebraically* less than zero; for $0-(-2) = +2$.

119. That a negative quantity is not *less* than zero in the arithmetical sense may be shown thus:

It is evident that $\frac{+1}{-1} = \frac{-1}{+1}$ (**117**). Now, if -1 is less than zero, much more will it be less than $+1$; that is, the numerator of the fraction $\frac{+1}{-1}$ will be greater than its denominator; hence, the numerator of the fraction $\frac{-1}{+1}$ will be greater than its denominator; therefore, -1 is less than $+1$ and greater than $+1$, which is absurd.

CHAPTER IV.

GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE.

GREATEST COMMON DIVISOR.

120. ***A Common Divisor*** or ***Common Measure*** of two or more quantities is any quantity that will exactly divide them. Thus, a, b, and ab are common divisors of ab^2 and abx. Any factor common to two or more quantities is a common divisor of them.

121. ***Commensurable Quantities*** are those which have a common divisor. Thus, ab^2 and abx are commensurable.

122. ***Incommensurable Quantities*** are those which have no common divisor. Thus, ab^2 and cdx are incommensurable.

123. ***The Greatest Common Divisor*** of two or more quantities is that common divisor of them which contains the greatest number of prime factors. Thus, $6a^2x$ is the greatest common divisor of $12a^2bx^2$ and $18a^3cxz$.

For brevity, we shall sometimes use G. C. D. for the phrase *greatest common divisor*.

124. To find the G. C. D. of two or more quantities.

Since every factor of a quantity is a divisor of that quantity, it follows that all the factors common to two or more quantities are all the common divisors of those quantities. Again, since the product of any number of factors of a quantity is a divisor of that quantity, it follows that the product of *all* the *factors common* to two or more quantities is *a common divisor* of those quantities.

Moreover, this product is the *greatest* common divisor, for it contains *all* the factors common to the given quantities; therefore, if another factor were introduced into this product, the result would not divide at least one of the given quantities.

Hence, when the given quantities can be resolved into prime factors by methods already explained, the G. C. D. may be found by the following

RULE.

Resolve each of the given quantities into its prime factors, then the product of all the prime factors which are common to those quantities will be the G. C. D. required.

EXAMPLES.

1. What is the G. C. D. of $4a^2bx$, $6ab^2x^3$, and $10a^2b^3c^4dx^2$?

$$4a^2bx = 2 \cdot 2 \cdot aabx,$$
$$6ab^2x^3 = 3 \cdot 2 \cdot abbxxx,$$
and
$$10a^2b^3c^4dx^2 = 2 \cdot 5aabbbccccdxx.$$

The common factors are 2, a, b, and x; hence, $2abx$ is the G. C. D. required. That $2abx$ is the *greatest* C. D. is evident; for if an additional factor, as a, be introduced, the product $2a^2bx$ will not be a divisor of *all* the given quantities.

2. What is the G. C. D. of $4am^2 + 4bm^2$ and $3an + 3bn$?

$$4am^2 + 4bm^2 = 2 \cdot 2 \cdot mm\,(a + b),$$
and
$$3an + 3bn = 3 \cdot n\,(a + b).$$

The only factor common to both the given quantities is $a + b$; hence, it is the G. C. D. required.

REMARK.—When there is only *one* common divisor, as in the preceding example, it would seem to be improper to speak of it as the *greatest* C. D. Nevertheless, since the *common* divisor, in such cases, is found by the general rule, we shall, for the sake of uniformity, call it the G. C. D.

3. What is the G. C. D. of $x^3 - y^3$ and $x^2 - y^2$?

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2),$$
and
$$x^2 - y^2 = (x - y)(x + y);$$

hence, $x - y$ is the G. C. D.

4. What is the G. C. D. of $ax^2 + 3ax + 2a$ and $ax^2 - ax - 2a$?

$$ax^2 + 3ax + 2a = a(x+1)(x+2),$$

and
$$ax^2 - ax - 2a = a(x+1)(x-2);$$

hence, $a(x+1)$ is the G. C. D.

5. What is the G. C. D. of $x^2 - 7x + 12$ and $x^2 - 8x + 15$?
Ans. $x - 3$.

6. What is the G. C. D. of $x^2 - x - 12$ and $ax^2 - 7ax + 12a$?
Ans. $x - 4$.

7. What is the G. C. D. of $2x^2 + 6x - 8$ and $2x^2 + 2x - 24$?
Ans. $2(x+4)$.

8. What is the G. C. D. of $2x^2 + 4xy + 2y^2$ and $3ax^2 + 6axy + 3ay^2$?
Ans. $(x+y)(x+y)$.

125. It is sometimes very difficult, if not impossible, to resolve the given quantities into their prime factors by inspection. We shall therefore proceed to demonstrate the following rule, which is more general in its application:

RULE FOR FINDING THE G. C. D. OF TWO ALGEBRAIC EXPRESSIONS.

I. *Let A and B denote the two expressions; let them be arranged according to the descending powers of some common letter, and suppose the exponent of the highest power of that letter in A not less than the exponent of the highest power of the same letter in B.*

II. *Divide A by B; then make the remainder a divisor and B the dividend. Again, make the new remainder a divisor and the preceding divisor the dividend. Proceed in this way until there is no remainder; then the last divisor is the G. C. D. required.*

The demonstration of the preceding rule depends upon the following Lemmas:

LEM. I.—If P is a divisor of A, then it will be a divisor of mA. For, since P is a divisor of A, we may suppose $A = a$P; then $m\text{A} = ma\text{P}$ (**42**, 4); but P is a divisor of maP; therefore, since $m\text{A} = ma\text{P}$, P is a divisor of mA.

LEM. II.—If P is a divisor of A and B, then it will be a divisor of $m\text{A} \pm n\text{B}$. For, since P is a divisor of A and B, we may suppose $\text{A}=a\text{P}$, and $\text{B}=b\text{P}$; then $m\text{A}\pm n\text{B}=(ma\pm nb)\text{P}$; hence, P is a divisor of $m\text{A} \pm n\text{B}$.

Let A and B denote the two expressions whose G. C. D. is to be found; let them be arranged according to the descending powers of some common letter, and suppose the exponent of the highest power of that letter in A not less than the exponent of the highest power of the same letter in B. Divide A by B; let p denote the quotient, and C the remainder. Divide B by C; let q denote the quotient, and D the remainder. Divide C by D; let r denote the quotient, and suppose there is no remainder.

$$\begin{array}{r|l}
\text{A} & \text{B} \\
p\text{B} & p \\
\hline
\end{array}\quad
\begin{array}{r|l}
\text{B} & \text{C} \\
q\text{C} & q \\
\hline
\end{array}\quad
\begin{array}{r|l}
\text{C} & \text{D} \\
r\text{D} & r \\
\hline
\end{array}$$

Now, since the dividend is equal to the product of the divisor and quotient, increased by the remainder, we have the three following equations:

$$\text{A}=p\text{B}+\text{C} \quad . \quad . \quad . \quad (1),$$

$$\text{B}=q\text{C}+\text{D} \quad . \quad . \quad . \quad (2),$$

$$\text{C}=r\text{D} \quad . \quad . \quad . \quad (3).$$

We shall first show that D is *a* common divisor of A and B. D is a divisor of C, since $\text{C}=r\text{D}$; hence (Lem. I), D is a divisor of $q\text{C}$, and, therefore (Lem. II), it is a divisor of $q\text{C}+\text{D}$; that is, D is a divisor of B. Again, since D is a divisor of B and C, it is a divisor of $p\text{B}+\text{C}$; that is, D is a divisor of A. Hence, D is a divisor of A and B.

We have thus shown that D is *a* common divisor of A and B; we shall next show that it is their *greatest* common divisor.

Equations (1) and (2) may be written as follows:

$$\text{A}-p\text{B}=\text{C} \quad . \quad . \quad . \quad (4),$$

$$\text{B}-q\text{C}=\text{D} \quad . \quad . \quad . \quad (5) \quad (\mathbf{42}, 3).$$

Now, *every* common divisor of A and B is a divisor of A—pB, that is, C (Lem. II); hence, *every* common divisor of A and B is a common divisor of B and C. Similarly, every common divi-

sor of B and C is a common divisor of C and D. We have thus shown that D is *a* common divisor of A and B, and that *every* common divisor of A and B is a divisor of D. But no expression of a higher degree than D is a divisor of D. Therefore, D is the G. C. D. required.

COR. 1.—Every common divisor of A and B is a divisor of their G. C. D.; and every divisor of their G. C. D. is a common divisor of A and B.

COR. 2.—Suppose we have to find the G. C. D. of A and B; and at any stage of the process suppose we have the expressions K and R, one of which is to be a dividend and the other a divisor. Let $R = mS$, where m has no factor which K has; then m may be rejected; that is, instead of continuing the process with K and R, we may continue it with K and S. For, by what has been already shown, we know that A and B have the same common divisors as K and R have. Now, any common divisor of K and S is a common divisor of K and R. Therefore, any common divisor of K and S is a common divisor of A and B. Again, any common divisor of K and R is a common divisor of K and mS, for $mS = R$. But m has no factor which K has. Therefore, any common divisor of K and R is a common divisor of K and S. Hence, A and B have the same common divisors as K and S have.

COR. 3.—A factor of a certain kind may be *introduced* at any stage of the process.

Suppose we have to find the G. C. D. of A and B; and at any stage of the process suppose we have the expressions K and R, one of which is to be a dividend and the other a divisor. Let $L = nK$, where n has no factor which R has; then n may be introduced; that is, instead of continuing the process with K and R, we may continue it with L and R. For A and B have the same common divisors as K and R have; and any common divisor of K and R is a common divisor of L and R. Therefore, any common divisor of A and B is a common divisor of L and R. Again, any common divisor of L and R is a common divisor of nK and R. But n has no factor that R has. Therefore, any

common divisor of L and R is a common divisor of K and R. Hence, A and B have the same common divisors as L and R have.

ILLUSTRATIONS.

1. Find the G. C. D. of x^2-6x+8 and $4x^3-21x^2+15x+20$. The operation may be arranged thus:

$$\begin{array}{r|l}
4x^3-21x^2+15x+20 & x^2-6x+8 \\ \cline{2-2}
4x^3-24x^2+32x\phantom{{}+20} & 4x+3 \\ \hline
3x^2-17x+20 & \\
3x^2-18x+24 & \\ \hline
\end{array}$$

$$\begin{array}{r|l}
x^2-6x+8 & x-4 \\ \cline{2-2}
x^2-4x\phantom{{}+8} & x-2 \\ \hline
-2x+8 & \\
-2x+8 & \\ \hline
\end{array}$$

Hence, $x-4$ is the G. C. D. required.

2. Find the G. C. D. of x^2+5x+4 and x^3+4x^2+5x+2.

$$\begin{array}{r|l}
x^3+4x^2+5x+2 & x^2+5x+4 \\ \cline{2-2}
x^3+5x^2+4x\phantom{{}+2} & x-1 \\ \hline
-x^2+x+2 & \\
-x^2-5x-4 & \\ \hline
\end{array}$$

$$\begin{array}{r|l}
x^2+5x+4 & 6x+6 \\ \cline{2-2}
x^2+x\phantom{{}+4} & \frac{x}{6}+\frac{4}{6} \\ \hline
4x+4 & \\
4x+4 & \\ \hline
\end{array}$$

This example introduces a new point for consideration. The last divisor here is $6x+6$; this, according to the rule, must be the G. C. D. required. When x^2+5x+4 is divided by $6x+6$, the quotient is $\frac{x}{6}+\frac{4}{6}$. If the other given expression be divided by $6x+6$, the quotient will be $\frac{x^2}{6}+\frac{x}{2}+\frac{1}{3}$.

It may at first appear that $6x+6$ cannot be a divisor of the two given expressions, since the quotients contain fractions. But we observe that in these quotients the letter x does not appear in

the denominator of any fraction. Such expressions as $\frac{x}{6} + \frac{4}{6}$ and $\frac{x^2}{6} + \frac{x}{2} + \frac{1}{3}$ are said to be *entire* with reference to x.

When we say that $6x + 6$ is the G. C. D. of the two given expressions, we mean that no common divisor can be found which contains a higher power of x than $6x + 6$. Other common divisors may be found which differ from this so far as respects numerical coefficients only. Thus, $3x + 3$ and $2x + 2$ are common divisors. Again, $x + 1$ is also a common divisor, and the corresponding quotients are $x + 4$ and $x^2 + 3x + 2$. We may then conveniently take $x + 1$ as the G. C. D., since the quotients do not contain fractional coefficients.

We may avoid fractional coefficients by proceeding as in the following example:

3. Find the G. C. D. of $3x^5 - 10x^3 + 15x + 8$ and $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6$.

$$
\begin{array}{r|l}
3x^5 - 10x^3 + 15x + 8 & x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6 \\ \cline{2-2}
3x^5 - 6x^4 - 18x^3 + 12x^2 + 39x + 18 & 3 \\ \cline{1-1}
6x^4 + 8x^3 - 12x^2 - 24x - 10 &
\end{array}
$$

Before proceeding to the next division, we may reject the factor 2 from every term of the new divisor (**125**, Cor. 2), and multiply every term of the new dividend by 3 (**125**, Cor. 3). We then continue the operation thus:

$$
\begin{array}{r|l}
3x^5 - 6x^4 - 18x^3 + 12x^2 + 39x + 18 & 3x^4 + 4x^3 - 6x^2 - 12x - 5 \\ \cline{2-2}
3x^5 + 4x^4 - 6x^3 - 12x^2 - 5x & x \\ \cline{1-1}
-10x^4 - 12x^3 + 24x^2 + 44x + 18 &
\end{array}
$$

Rejecting the factor 2 from every term of the last remainder, and multiplying the result by 3, we have the expression,

$$- 15x^4 - 18x^3 + 36x^2 + 66x + 27.$$

We then continue the operation thus:

$$
\begin{array}{r|l}
- 15x^4 - 18x^3 + 36x^2 + 66x + 27 & 3x^4 + 4x^3 - 6x^2 - 12x - 5 \\ \cline{2-2}
- 15x^4 - 20x^3 + 30x^2 + 60x + 25 & - 5 \\ \cline{1-1}
2x^3 + 6x^2 + 6x + 2 &
\end{array}
$$

We now reject the factor 2 from every term of this remainder, and continue the operation thus:

$$\begin{array}{l|l} 3x^4 + 4x^3 - 6x^2 - 12x - 5 & x^3 + 3x^2 + 3x + 1 \\ \underline{3x^4 + 9x^3 + 9x^2 + 3x} & \overline{3x - 5} \\ - 5x^3 - 15x^2 - 15x - 5 & \\ \underline{- 5x^3 - 15x^2 - 15x - 5} & \end{array}$$

Hence, $x^3 + 3x^2 + 3x + 1$ is the G. C. D. required.

126. Suggestions.—Suppose the given expressions A and B to contain a common factor F, which is obvious on inspection. Let $A = aF$, and $B = bF$. Then F will be a factor of the G. C. D. (**120**). We may then find the G. C. D. of a and b, and multiply it by F; the product will be the G. C. D. of A and B.

In like manner, if at any stage of the operation we perceive that a certain factor is common to the dividend and divisor, we may omit it and continue the operation with the remaining factors. The factor omitted must then be multiplied by the last divisor obtained by continuing the operation; the product will be the G. C. D. required.

127. To find the G. C. D. of three Algebraic Expressions, A, B, and C.

Find the G. C. D. of two of them, as A and B. Let D denote this G. C. D.; then the G. C. D. of C and D will be the G. C. D. of A, B, and C. For every common divisor of C and D is a common divisor of A, B, and C (**125,** Cor. 1). Again, every common divisor of A, B, and C is a common divisor of C and D. Hence, the G. C. D. of C and D is the G. C. D. of A, B, and C.

EXAMPLES.

128. Find the G. C. D. of

1. $x^2 - 3x + 2$ and $x^2 - x - 2$. *Ans.* $x - 2$.
2. $x^3+3x^2+4x+12$ and x^3+4x^2+4x+3. *Ans.* $x + 3$.
3. x^3+x^2+x-3 and x^3+3x^2+5x+3. *Ans.* $x^2 + 2x + 3$.

4. $x^3 + 1$ and $x^3 + mx^2 + mx + 1$. *Ans.* $x + 1$.

5. $6x^3 - 7ax^2 - 20a^2x$ and $3x^2 + ax - 4a^2$. *Ans.* $3x + 4a$.

6. $x^5 - y^5$ and $x^2 - y^2$. *Ans.* $x - y$.

7. $3x^3 - 13x^2 + 23x - 21$ and $6x^3 + x^2 - 44x + 21$.
Ans. $3x - 7$.

8. $x^4 - 3x^3 + 2x^2 + x - 1$ and $x^3 - x^2 - 2x + 2$.
Ans. $x - 1$.

9. $x^4 - 7x^3 + 8x^2 + 28x - 48$ and $x^3 - 8x^2 + 19x - 14$.
Ans. $x - 2$.

10. $x^4 - x^3 + 2x^2 + x + 3$ and $x^4 + 2x^3 - x - 2$.
Ans. $x^2 + x + 1$.

11. $4x^4 + 9x^3 + 2x^2 - 2x - 4$ and $3x^3 + 5x^2 - x + 2$.
Ans. $x + 2$.

12. $2x^4 - 12x^3 + 19x^2 - 6x + 9$ and $4x^3 - 18x^2 + 19x - 3$.
Ans. $x - 3$.

13. $6x^4 + x^3 - x$ and $4x^3 - 6x^2 - 4x + 3$. *Ans.* $2x - 1$.

14. $2x^4 + 11x^3 - 13x^2 - 99x - 45$ and $2x^3 - 7x^2 - 46x - 21$.
Ans. $2x^2 + 7x + 3$.

15. $x^3 - 9x^2 + 26x - 24$, $x^3 - 10x^2 + 31x - 30$, and $x^3 - 11x^2 + 38x - 40$. *Ans.* $x - 2$.

LEAST COMMON MULTIPLE.

129. When one quantity is divisible by another, the first is called a ***Multiple*** of the other. Thus, 6 is a multiple of 2, and ab is a multiple of b.

130. ***A Common Multiple*** of two or more quantities is a quantity which is divisible by each of them. Thus, 12 is a common multiple of 2 and 3, and $20xy$ is a common multiple of $2x$ and $5y$.

131. ***The Least Common Multiple*** of two or more quantities is that common multiple of them which contains the least number of prime factors. Thus, 6 is the least common multiple of 2 and 3, and $10xy$ is the least common multiple of $2x$ and $5y$.

For brevity, we shall sometimes use L. C. M. for the phrase *least common multiple.*

132. To find the L. C. M. of two or more quantities.

It is obvious that the L. C. M. of two or more quantities must contain *all* the factors of each of them, and *no other* factors. Hence, when the given quantities can be readily resolved into their prime factors, the L. C. M. may be found by the following

RULE.

I. *Resolve each of the given quantities into its prime factors.*

II. *Multiply one of the given quantities by the product of such prime factors of the other quantities as are not found in it; the result will be the L. C. M. required.*

COR.—If the given quantities are relatively prime (**90**), their product is their L. C. M. Thus, the L. C. M. of $7ab$ and $6cd$ is $42abcd$.

EXAMPLES.

1. Find the L. C. M. of $9x^2y$ and $12xy^2$.

$$9x^2y = 3 \cdot 3 \cdot x \cdot x \cdot y, \quad \text{and} \quad 12xy^2 = 3 \cdot 2 \cdot 2 \cdot x \cdot y \cdot y;$$

hence, the L. C. M. is $9x^2y \times 2 \cdot 2 \cdot y = 36x^2y^2$.

2. Find the L. C. M. of $4a^2b^2$, $6a^2b$, and $10a^3x^2$.

$$4a^2b^2 = 2 \cdot 2a^2b^2,\ 6a^2b = 2 \cdot 3a^2b, \text{ and } 10a^3x^2 = 2 \cdot 5a^3x^2;$$

hence, the L. C. M. is $4a^2b^2 \times 3 \times 5ax^2 = 60a^3b^2x^2$.

It is not necessary, when the given quantities are monomials, to actually separate the literal parts into prime factors, since the exponent of any letter shows how many times it occurs as a factor.

3. Find the L. C. M. of $a^2x - 2abx + b^2x$ and $a^2y - b^2y$.

$$a^2x - 2abx + b^2x = (a - b)(a - b)x, \text{ and } a^2y - b^2y = (a + b)(a - b)y;$$

hence, the L. C. M. is $(a^2x - 2abx + b^2x)(a + b)y$.

4. Find the L. C. M. of $2a^3b^2cx$, $3a^5bc^3x^2$, $6acx$, $9c^7x^{10}$, and $24a^3$. *Ans.* $72a^5b^2c^7x^{10}$.

5. Find the L. C. M. of $16ax$, $40b^5x$, and $25a^7b^3x^2$. *Ans.* $400a^7b^5x^2$.

6. Find the L. C. M. of $x^2 - 3x + 2$ and $x^2 - 1$. *Ans.* $x^3 - 2x^2 - x + 2$.

7. Find the L. C. M. of $a^3x + b^3x$ and $a^2 - b^2$. *Ans.* $a^4x - a^3bx + ab^3x - b^4x$.

8. Find the L. C. M. of $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$. *Ans.* $(a^2 - b^2)^2$.

9. Find the L. C. M. of $a^2 + b^2$ and $a^2 - b^2$. *Ans.* $a^4 - b^4$.

10. Find the L. C. M. of $x^3 - x$ and $x^2 - 1$. *Ans.* $x^3 - x$.

11. Find the L. C. M. of $xz + yz$ and $x^2y + xy^2$. *Ans.* $x^2yz + xy^2z$.

133. It is sometimes very difficult, if not impossible, to resolve the given quantities into their prime factors by inspection. We shall therefore proceed to demonstrate the following rule, which is more general in its application:

RULE FOR FINDING THE L. C. M. OF TWO ALGEBRAIC EXPRESSIONS.

Divide the product of the two expressions by their G. C. D.; or divide one of the expressions by the G. C. D. and multiply the quotient by the other expression.

Let A and B denote the two expressions, and D their G. C. D. Suppose $A = aD$, and $B = bD$. From the nature of the G. C. D., a and b have no common factor; hence, the L. C. M. of A and B is abD. But $abD = \frac{AB}{D} = \frac{A}{D} \times B = \frac{B}{D} \times A$.

Cor.—If M be the L. C. M. of A and B, it is obvious that every multiple of M is a common multiple of A and B.

134. *Every common multiple of two algebraic expressions is a multiple of their L. C. M.*

Let A and B denote the two expressions, M their L. C. M., and let N denote any other common multiple. Suppose, if possible, that when N is divided by M, there is a remainder, R; let q denote the quotient. Then $R = N - qM$. Now A and B are common divisors of M and N, and therefore they are divisors of R (**125,** Lem. II); that is, R is a common multiple of A and B. But R is of *lower* dimensions than M; hence, there is a common multiple of A and B of lower dimensions than their L. C. M. This is absurd; hence, there can be no remainder; that is, N is a multiple of M.

135. To find the L. C. M. of three Algebraic expressions, A, B, and C.

Find the L. C. M. of two of them, as A and B. Let M denote this L. C. M.; then the L. C. M. of M and C is the required L. C. M. of A, B, and C.

For every common multiple of M and C is a common multiple of A, B, and C (**133,** Cor.). Again, every common multiple of A, B, and C is a multiple of M and C (**134**). Therefore, the L. C. M. of M and C is the L. C. M. of A, B, and C.

EXAMPLES.

136. Find the L. C. M. of

1. $6x^2 - x - 1$ and $2x^2 + 3x - 2$.
   Ans. $(2x^2 + 3x - 2)(3x + 1)$.
2. $x^3 - 1$ and $x^2 + x - 2$. *Ans.* $(x^3 - 1)(x + 2)$.
3. $x^3 - 9x^2 + 23x - 15$ and $x^2 - 8x + 7$.
   Ans. $(x^3 - 9x^2 + 23x - 15)(x - 7)$.
4. $3x^2 - 5x + 2$ and $4x^3 - 4x^2 - x + 1$.
   Ans. $(3x - 2)(4x^3 - 4x^2 - x + 1)$.
5. $(x + 1)(x^2 - 1)$ and $x^3 - 1$. *Ans.* $(x^3 - 1)(x + 1)^2$.
6. $x^3 + 2x^2y - xy^2 - 2y^3$ and $x^3 - 2x^2y - xy^2 + 2y^3$.
   Ans. $(x^2 - y^2)(x^2 - 4y^2)$.

7. $2x - 1$, $4x^2 - 1$, and $4x^2 + 1$. *Ans.* $16x^4 - 1$.

8. $x^3 - x$, $x^3 - 1$, and $x^3 + 1$. *Ans.* $x(x^6 - 1)$.

9. $x^2 - 4a^2$, $(x + 2a)^3$, and $(x - 2a)^3$. *Ans.* $(x^2 - 4a^2)^3$.

10. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$, and $x^3 - 8x^2 + 19x - 12$. *Ans.* $(x - 1)(x - 2)(x - 3)(x - 4)$.

11. $x^2 + 7x + 10$, $x^2 - 2x - 8$, and $x^2 + x - 20$.
Ans. $x^3 + 3x^2 - 18x - 40$.

12. $a^2 - 3ab + 2b^2$, $a^2 - ab - 2b^2$, and $a^2 - b^2$.
Ans. $a^3 - 2a^2b - ab^2 + 2b^2$.

13. $2x^2 - 7xy + 3y^2$, $2x^2 - 5xy + 2y^2$, and $x^2 - 5xy + 6y^2$.
Ans. $2x^3 - 11x^2y + 17xy^2 - 6y^3$.

137. SYNOPSIS FOR REVIEW.

CHAPTER IV.

- G. C. D. . . . .
  - TERMS USED . . . . . .
    - *Common Divisor.*
    - *G. C. D.*
    - *Commensurable.*
    - *Incommensurable.*
  - SPECIAL RULE—DEM.
  - GENERAL RULE.—DEM. COR. 1, COR. 2, COR. 3.
    - *Lemma I—Dem.*
    - *Lemma II—Dem.*
    - *Application of Lemmas.*
  - SUGGESTIONS . . . . . .
    - *Introduction of Factors.*
    - *Rejection of Factors.*
  - G. C. D. OF THREE ALGEBRAIC EXPRESSIONS.
- L. C. M. . . . .
  - TERMS USED . . . . . .
    - *Multiple.*
    - *Common Multiple.*
    - *L. C. M.*
  - SPECIAL RULE—DEM.—COR.
  - GENERAL RULE—DEM.
  - THEOREM—DEM.
  - L. C. M. OF THREE ALGEBRAIC EXPRESSIONS.

CHAPTER V.

FRACTIONS.

DEFINITIONS AND FUNDAMENTAL PRINCIPLES.

138. ***A Fraction*** is a quotient expressed by placing the dividend over the divisor, with a line between them. Thus, $\frac{3}{4}$ and $\frac{a}{b}$ are fractions.

139. ***The Numerator*** of a fraction is the quantity above the line, and the ***Denominator*** is the quantity below the line. The numerator and denominator of a fraction are called its *Terms*.

140. If the terms of a fraction are integers, we may regard the denominator as denoting the number of equal parts into which the unit (1) is divided, and the numerator as denoting how many of those parts are expressed.

141. ***A Fractional Unit*** is one of the equal parts into which the unit is divided. Thus, in the fraction $\frac{a}{b}$, the fractional unit is $\frac{1}{b}$.

142. An integer may be considered as a fraction with unity for its denominator. Thus, $a = \frac{a}{1}$.

143. ***An Entire Quantity*** is one which does not contain a fraction. Thus, $a + b + c$ is an entire quantity.

144. ***A Mixed Quantity*** is one which contains an en-

tire part and a fractional part. Thus, $a+b+\frac{c}{d}$ is a mixed quantity.

145. ***A Simple Fraction*** is one whose terms are entire. Thus, $\frac{a+b}{c+d}$ is a simple fraction.

146. ***A Complex Fraction*** is one which has a fraction in one or both of its terms. Thus, $\dfrac{a+\frac{b}{c}}{d+\frac{e}{f}}$ is a complex fraction.

147. ***A Compound Fraction*** is the indicated product of two or more fractions. Thus, $\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}$ is a compound fraction.

148. ***A Proper Fraction*** is one whose numerator is less than its denominator. Thus, $\frac{a}{a+b}$ is a proper fraction.

149. ***An Improper Fraction*** is one whose numerator is equal to or greater than its denominator. Thus, $\frac{a}{a}$ and $\frac{a+b}{a}$ are improper fractions.

150. ***The Reciprocal*** of a quantity is the quotient obtained by dividing unity by that quantity. Thus, the reciprocal of a is $\frac{1}{a}$.

151. To multiply a fraction by an integer.

Let $\frac{a}{b}$ be a fraction, and c an integer; then $\frac{a}{b}\times c=\frac{ac}{b}$.

For, in each of the fractions $\frac{a}{b}$ and $\frac{ac}{b}$ the fractional unit (**141**) is $\frac{1}{b}$; hence, $\frac{ac}{b}$ is c times $\frac{a}{b}$ (**140**).

Again, $\frac{a}{bc}\times c=\frac{a}{b}$. For, the fractional unit in $\frac{a}{b}$ is c times the fractional unit in $\frac{a}{bc}$.

RULE.

Multiply the given numerator by the given integer, and divide the product by the given denominator, or, divide the given denominator by the given integer, and divide the given numerator by the quotient.

152. To divide a fraction by an integer.

$$\frac{a}{b} \div c = \frac{a}{bc}.$$

For, $\frac{a}{b}$ is c times $\frac{a}{bc}$ (**151**); hence, $\frac{a}{bc}$ is $\frac{1}{c}$th of $\frac{a}{b}$.

Again, $$\frac{ac}{b} \div c = \frac{a}{b}$$

For, $\frac{ac}{b}$ is c times $\frac{a}{b}$; hence, $\frac{a}{b}$ is $\frac{1}{c}$th of $\frac{ac}{b}$.

RULE.

Multiply the given denominator by the given integer, and divide the given numerator by the product, or, divide the given numerator by the given integer, and divide the quotient by the given denominator.

153. *The value of a fraction is not changed by multiplying or dividing both of its terms by the same quantity.*

It is evident that if we *multiply* the fraction $\frac{a}{b}$ by c, and then *divide* the product by c, the resulting fraction will be equal to the given fraction. Now $\frac{a}{b} \times c = \frac{ac}{b}$ (**151**), and $\frac{ac}{b} \div c = \frac{ac}{bc}$ (**152**); hence, $$\frac{a}{b} = \frac{ac}{bc}, \text{ or } \frac{ac}{bc} = \frac{a}{b}.$$

REDUCTION OF FRACTIONS.

154. A fraction is in its ***Lowest Terms*** when its terms have no common factor.

155. To reduce a fraction to its lowest terms.

RULE.

Divide both terms of the fraction by their G. C. D.

Or, *Resolve both terms of the fraction into their prime factors, and then cancel those factors which are common.*

ILLUSTRATIONS.

1. Reduce $\frac{10acx^2}{15bcx^3}$ to its lowest terms.

The G. C. D. of the terms of this fraction is $5cx^2$.
Dividing both terms by this, we have

$$\frac{10acx^2}{15bcx^3} = \frac{2a}{3bx}.$$

2. Reduce $\frac{3a^2 + 3ab}{3a^2 - 3ab}$ to its lowest terms.

$$\frac{3a^2 + 3ab}{3a^2 - 3ab} = \frac{3a(a+b)}{3a(a-b)} = \frac{a+b}{a-b}.$$

3. Reduce $\frac{6x^2 - 7x - 20}{4x^3 - 27x + 5}$ to its lowest terms.

Here the G. C. D. of the numerator and denominator is $2x - 5$. Dividing both terms of the fraction by this, we have

$$\frac{6x^2 - 7x - 20}{4x^3 - 27x + 5} = \frac{3x + 4}{2x^2 + 5x - 1}.$$

156. To reduce a fraction to an entire or mixed quantity.

$$\frac{a^2 + ab}{a} = a + b, \quad \frac{a^2 + b}{a} = a + \frac{b}{a}, \quad \text{and} \quad \frac{a^2 - b}{a} = a - \frac{b}{a}.$$

RULE.

Divide the numerator by the denominator, expressing any term of the quotient in a fractional form when the division cannot be exactly performed.

157. To reduce an entire quantity to the form of a fraction having a given denominator.

Let it be required to reduce $x + y$ to the form of a fraction whose denominator shall be $x - y$.

$$x + y = \frac{x + y}{1} = \frac{(x + y)(x - y)}{x - y} = \frac{x^2 - y^2}{x - y}.$$

RULE.

Consider the entire quantity as a fraction whose denominator is unity; then multiply both terms of this fraction by the given denominator.

158. To reduce a fraction to an equivalent one having a given denominator.

Let it be required to reduce the fraction $\frac{a + b}{a - b}$ to an equivalent one having the denominator $a^2 - b^2$.

Dividing $a^2 - b^2$ by $a - b$, the quotient is $a + b$.

Multiplying both terms of the fraction $\frac{a + b}{a - b}$ by this quotient, we have

$$\frac{a + b}{a - b} = \frac{(a + b)(a + b)}{a^2 - b^2} = \frac{(a + b)^2}{a^2 - b^2}.$$

RULE.

Multiply both terms of the given fraction by the quotient obtained by dividing the denominator of the required fraction by the denominator of the given fraction.

159. To reduce a mixed quantity to the form of a fraction.

Let it be required to express $a + \frac{b}{c}$ under a fractional form.

$$a + \frac{b}{c} = \frac{ac + b}{c}.$$

For, if we reduce the fraction $\frac{ac + b}{c}$ to a mixed quantity, we obtain $a + \frac{b}{c}$ (**156**).

In like manner we may show that $a - \frac{b}{c} = \frac{ac - b}{c}$.

RULE.

Multiply the entire part by the denominator of the fractional part; then add the numerator to the product, or subtract it from the product, according as the fraction has the sign +, or the sign —, prefixed; the result will be the numerator, and the given denominator will be the denominator of the required fraction.

160. To reduce fractions to equivalent ones having a common denominator.

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be the proposed fractions. If we multiply both terms of each of these fractions by the product of all the denominators except its own, the values of the fractions will not be changed (**153**). Moreover, the denominators of the new fractions will be equal, since each is the product of the denominators of the given fractions.

Thus, $\frac{a}{b} = \frac{adf}{bdf}$, $\frac{c}{d} = \frac{bcf}{bdf}$, and $\frac{e}{f} = \frac{bde}{bdf}$.

RULE.

Multiply both terms of each of the given fractions by the product of all the denominators except its own.

161. To reduce fractions to equivalent ones having the least common denominator.

Let $\frac{a}{mx}$, $\frac{b}{my}$, and $\frac{c}{mz}$ be the proposed fractions. The L. C. M. of the denominators is $mxyz$. Now reduce each of the given frac-

tions to an equivalent one having $mxyz$ for its denominator (**158**); the resulting fractions are

$$\frac{ayz}{mxyz}, \quad \frac{bxz}{mxyz}, \quad \text{and} \quad \frac{cxy}{mxyz}.$$

Now since $mxyz$ is the least quantity that can be divided separately by mx, my, and mz, it follows that the given fractions have been reduced to equivalent ones having the *least* common denominator.

RULE.

Divide the L. C. M. of all the denominators by each denominator separately; then multiply both terms of each fraction by the corresponding quotient.

SCH.—Before commencing the operation, each fraction must be in its lowest terms.

COMBINATIONS OF FRACTIONS.

162. To find the sum of given fractions.

1. Let it be required to find the sum of the fractions $\frac{a}{b}$, $\frac{c}{b}$, and $\frac{d}{b}$. Here the given fractions have a common denominator. In the first fraction the fractional unit $\frac{1}{b}$ is taken a times; in the second it is taken c times; and in the third, d times; hence, in the sum of these fractions $\frac{1}{b}$ must be taken $(a + c + d)$ times; therefore,

$$\frac{a}{b} + \frac{c}{b} + \frac{d}{b} = \frac{a + c + d}{b}.$$

2. Let it be required to find the sum of the fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$. Here the given fractions have unequal denominators. Reducing them to equivalent fractions having a common denominator (**160**), we have,

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf}{bdf} + \frac{bcf}{bdf} + \frac{bde}{bdf} = \frac{adf + bcf + bde}{bdf}.$$

RULES.

I. *If the given fractions have a common denominator, form a fraction whose numerator is the sum of the given numerators, and whose denominator is the given common denominator; this fraction will be the sum of the given fractions.*

II. *If the given fractions have not a common denominator, reduce them to equivalent ones having a common denominator; then proceed as directed in I.*

163. To find the difference between two fractions.

1. Let it be required to subtract $\frac{c}{b}$ from $\frac{a}{b}$. The fractional unit $\frac{1}{b}$ is taken a times in the minuend, and c times in the subtrahend; hence, it must be taken $(a - c)$ times in the remainder; therefore,

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$$

2. Let it be required to subtract $\frac{c}{d}$ from $\frac{a}{b}$.

Reducing these fractions to equivalent ones having a common denominator, we have

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}.$$

RULES.

I. *If the given fractions have a common denominator, form a fraction whose numerator is the remainder obtained by subtracting the numerator of the subtrahend from that of the minuend, and whose denominator is the given common denominator; this fraction will be the difference required.*

II. *If the given fractions have not a common denominator, reduce them to equivalent ones having a common denominator; then proceed as directed in I.*

164. To find the product of given fractions.

Let it be required to find the product of $\frac{a}{b}$ and $\frac{c}{d}$. The following is usually given as a solution:

Put $\frac{a}{b}=m$, and $\frac{c}{d}=n$.

Then $a=bm$, and $c=dn$.

Hence, $ac=bmdn=bd\times mn$; or, dividing both members by bd (**42**, 5), we have $\frac{ac}{bd}=mn$.

This process is satisfactory when m and n are *really integers*, though under a fractional form, because then the word *multiplication* has its common meaning. It is also satisfactory when *one* of them is an integer, because we can speak of multiplying a fraction by an integer, as in Art. **151.** But when both m and n are fractions, we cannot speak of multiplying one of them by the other without defining what we mean by the term *multiplication;* for, according to the ordinary meaning of this term, the *multiplier* must be an integer.

The following definitions will show more clearly the connection between the meaning of the word multiplication when applied to integers, and its meaning when applied to fractions. When we multiply one integer, a, by another, b, we may describe the operation thus:

What we did with unity to obtain b, we must now do with a to obtain b times a.

Now, let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$. Adopting the definition just given, we may say that, *what we did with unity to obtain* $\frac{c}{d}$, we must now do with $\frac{a}{b}$ to obtain the product of $\frac{a}{b}$ and $\frac{c}{d}$. To obtain $\frac{c}{d}$ from unity, we divide it into d equal parts, and multiply one of the parts by c; therefore, to obtain the product of $\frac{a}{b}$ and $\frac{c}{d}$, we divide $\frac{a}{b}$ into d equal parts, and multiply

one of them by c. Now $\frac{a}{b} \div d = \frac{a}{bd}$ **(152)**, and $\frac{a}{bd} \times c = \frac{ac}{bd}$ **(151)**.

We may therefore give the following extended definition:

Multiplication is the process of finding a quantity having the same relation to the multiplicand that the multiplier has to unity.

RULE.

Form a fraction whose numerator is the product of the given numerators, and whose denominator is the product of the given denominators; this fraction will be the product required.

SCH. 1.—This rule embraces all the cases in which a fraction is a factor. Thus, if it be required to multiply a fraction by an entire quantity, the latter may be considered as a fraction whose denominator is unity **(142)**.

SCH. 2.—If any factor is a mixed quantity, it is best to reduce it to the form of a fraction before commencing the operation.

SCH. 3.—If the numerator and denominator of the product have any common factor, it should be canceled. Thus,

$$\frac{2a^2}{a^2-b^2} \times \frac{(a+b)^2}{4a^2b} = \frac{2a^2(a+b)^2}{(a^2-b^2)4a^2b} = \frac{2a^2(a+b)(a+b)}{4a^2b(a+b)(a-b)} = \frac{a+b}{2b(a-b)}$$

(155).

165. To find the quotient of two fractions.

Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$. Denoting the quotient by x, we have

$$\frac{a}{b} \div \frac{c}{d} = x.$$

But the product of the divisor and quotient is equal to the dividend; hence,

$$x \times \frac{c}{d} = \frac{a}{b}.$$

Multiplying both members of this equation by $\frac{d}{c}$ **(42**, 4), we have

$$x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}.$$

Canceling common factors (**155**), we have

$$x = \frac{a}{b} \times \frac{d}{c};$$

that is, the quotient is equal to the product obtained by multiplying the dividend by the divisor inverted.

RULE.

Multiply the dividend by the divisor inverted.

COR. 1.—The product of a quantity and its reciprocal is unity. Thus, $a \times \frac{1}{a} = 1$.

COR. 2.—To divide by a quantity is the same as to multiply by its reciprocal; and, conversely, to multiply by a quantity is the same as to divide by its reciprocal. Thus,

$$a \div b = a \times \frac{1}{b}, \quad \text{and} \quad a \times b = a \div \frac{1}{b}.$$

166. In the present chapter we have thus far supposed each letter to represent an *integer;* but, by virtue of our extended definitions, it may be shown that all the rules and formulæ given are true when any letter represents a fraction. For example, let it be required to show that $\frac{a}{b} = \frac{ac}{bc}$ when $a = \frac{m}{n}$, $b = \frac{p}{q}$, and $c = \frac{r}{s}$.

$$\frac{a}{b} = \frac{m}{n} \div \frac{p}{q} = \frac{m}{n} \times \frac{q}{p} = \frac{mq}{np},$$

$$ac = \frac{m}{n} \times \frac{r}{s} = \frac{mr}{ns},$$

$$bc = \frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs};$$

hence, $$\frac{ac}{bc} = \frac{mr}{ns} \div \frac{pr}{qs} = \frac{mr}{ns} \times \frac{qs}{pr} = \frac{mrqs}{nspr} = \frac{mq}{np}.$$

THE SIGNS OF FRACTIONS.

167. *Each sign* in the numerator and denominator of a fraction affects only the term to which it is prefixed. Thus, in the fraction $\frac{a-b}{c-d}$, the sign of a is $+$, that of b is $-$, that of c is $+$, and that of d is $-$.

168. *The dividing line* of a fraction answers the purpose of a *vinculum;* that is, it connects the terms which the numerator and denominator may each contain. Therefore the sign prefixed to the dividing line affects the fraction as a *whole.*

169. If the sign prefixed to the dividing line be changed, the sign of the fraction will be changed. Thus, $\frac{ab}{b}=a$; but $-\frac{ab}{b}=-a$.

170. If the sign of each term of the numerator be changed, the sign of the fraction will be changed. Thus, $\frac{ab-bc}{b}=a-c$; but $\frac{-ab+bc}{b}=-a+c$.

171. If the sign of each term of the denominator be changed, the sign of the fraction will be changed. Thus, $\frac{ab}{b}=a$; but $\frac{ab}{-b}=-a$.

172. We may sum up the three preceding Articles thus:

If the sign prefixed to a fraction, or the sign of each term of the numerator, or the sign of each term of the denominator, be changed, the sign of the fraction will be changed.

Cor.—If any two of these changes be made at the same time, the sign of the fraction will not be changed.

173. ***The Apparent Sign*** of a fraction is the sign prefixed to the dividing line of that fraction. ***The Real Sign***

of a fraction is the sign of its numerical value. Thus, the apparent sign of the fraction $-\frac{a-b}{c}$ is $-$; but, if $a=3$, $b=4$, and $c=5$, the real sign is $+$.

174. *EXAMPLES.*

Simplify the following fractions, from 1 to 12, inclusive:

1. $\frac{x^2+2x-3}{x^2+6x-7}$. *Ans.* $\frac{x+3}{x+7}$.

2. $\frac{x^2-3x-4}{x^2-4x-5}$. *Ans.* $\frac{x-4}{x-5}$.

3. $\frac{x^3-6x^2+11x-6}{x^2-3x+2}$. *Ans.* $x-3$.

4. $\frac{a^3+3a^2b+3ab^2+b^3}{a^2+2ab+b^2}$. *Ans.* $a+b$.

5. $\frac{x^4+10x^3+35x^2+50x+24}{x^3+9x^2+26x+24}$. *Ans.* $x+1$.

6. $\frac{3x^3-16x^2+23x-6}{2x^3-11x^2+17x-6}$. *Ans.* $\frac{3x-1}{2x-1}$.

7. $\frac{6x^3-5x^2+4}{2x^3-x^2-x+2}$. *Ans.* $\frac{3x+2}{x+1}$.

8. $\frac{2x^3+9x^2+7x-3}{3x^3+5x^2-15x+4}$. *Ans.* $\frac{2x+3}{3x-4}$.

9. $\frac{1-x}{1-x^2}$. *Ans.* $\frac{1}{1+x}$.

10. $\frac{5a^2+5ax}{a^2-x^2}$. *Ans.* $\frac{5a}{a-x}$.

11. $\frac{x^4+2x^2+9}{x^4-4x^3+4x^2-9}$. *Ans.* $\frac{x^2+2x+3}{x^2-2x-3}$.

12. $\frac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}$. *Ans.* $\frac{x+a}{x+b}$.

Perform the additions and subtractions indicated in the following examples, from 13 to 28 inclusive:

13. $\frac{a}{a+b}+\frac{b}{a-b}$ *Ans.* $\frac{a^2+b^2}{a^2-b^2}$.

14. $\frac{a}{2a-2b}+\frac{b}{2b-2a}$ *Ans.* $\frac{1}{2}$.

15. $\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}$. *Ans.* $-\frac{2}{(4x^2-1)x}$.

16. $\left(\frac{1}{m}+\frac{1}{n}\right)(a+b)-\left(\frac{a+b}{m}-\frac{a-b}{n}\right)$. *Ans.* $\frac{2a}{n}$.

17. $\frac{1}{x-1}-\frac{1}{x+2}-\frac{3}{(x+2)^2}$. *Ans.* $\frac{9}{(x-1)(x+2)^2}$.

18. $\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$.

Ans. $\frac{2x-3}{(x^2-1)(2x+3)}$.

19. $\frac{b-a}{x-b}-\frac{a-2b}{x+b}+\frac{3x(a-b)}{x^2-b^2}$ *Ans.* $\frac{ax-b^2}{x^2-b^2}$.

20. $\frac{3+2x}{2-x}-\frac{2-3x}{2+x}+\frac{16x-x^2}{x^2-4}$. *Ans.* $\frac{1}{x+2}$.

21. $\frac{3}{1-2x}-\frac{7}{1+2x}-\frac{4-20x}{4x^2-1}$. *Ans.* 0.

22. $\frac{1}{a+b}+\frac{b}{a^2-b^2}-\frac{a}{a^2+b^2}$. *Ans.* $\frac{2ab^2}{a^4-b^4}$.

23. $\frac{1}{x^2-y^2}+\frac{1}{(x+y)^2}-\frac{1}{(x-y)^2}$. *Ans.* $\frac{x^2-4xy-y^2}{(x^2-y^2)^2}$.

24. $\frac{(a^2+b^2)^2}{ab(a-b)^2}-\frac{a}{b}-\frac{b}{a}-2$. *Ans.* $\frac{4ab}{(a-b)^2}$.

25. $\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}$. *Ans.* $\frac{4a}{a+x}$.

26. $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}-\frac{a-4b}{21}$.

Ans. $\frac{81a-4b}{84}$.

27. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}$.

Ans. 0.

28. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ac}{(b+c)(b+a)}+\dfrac{c^2-ab}{(c+a)(c+b)}$.

Ans. 0.

29. Multiply $\dfrac{(a-b)^2}{a+b}$ by $\dfrac{b}{x(a-b)}$. *Ans.* $\dfrac{(a-b)b}{(a+b)x}$.

30. Multiply $\dfrac{x^2+xy}{x^2+y^2}$ by $\dfrac{x^3-y^3}{xy(x+y)}$. *Ans.* $\dfrac{x^3-y^3}{y(x^2+y^2)}$.

31. Multiply together $\dfrac{3ax}{4by}$, $\dfrac{a^2-x^2}{c^2-x^2}$, $\dfrac{bc+bx}{a^2+ax}$, and $\dfrac{c-x}{a-x}$.

Ans. $\dfrac{3x}{4y}$.

32. Prove that

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{a}{c}+\frac{c}{a}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

33. Multiply together $\dfrac{1-x^2}{1+y}$, $\dfrac{1-y^2}{x+x^2}$, and $1+\dfrac{x}{1-x}$.

Ans. $\dfrac{1-y}{x}$.

34. Multiply $\dfrac{x(a-x)}{a^2+2ax+x^2}$ by $\dfrac{a(a+x)}{a^2-2ax+x^2}$.

Ans. $\dfrac{ax}{a^2-x^2}$.

35. Simplify $\dfrac{a^4-b^4}{a^2-2ab+b^2}\times\dfrac{a-b}{a^2+ab}$. *Ans.* $\dfrac{a^2+b^2}{a}$.

36. Simplify $\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}\right)\dfrac{x+y}{2y}$. *Ans.* 2.

37. Simplify $\dfrac{a^3-b^3}{a^3+b^3}\cdot\dfrac{a+b}{a-b}\cdot\left(\dfrac{a^2-ab+b^2}{a^2+ab+b^2}\right)^2$.

Ans. $\dfrac{a^2-ab+b^2}{a^2+ab+b^2}$.

38. Multiply $\dfrac{x^2}{a^2}-\dfrac{x}{a}+1$ by $\dfrac{x^2}{a^2}+\dfrac{x}{a}+1$. *Ans.* $\dfrac{x^4}{a^4}+\dfrac{x^2}{a^2}+1$.

39. Multiply $x^2 - x + 1$ by $\frac{1}{x^2} + \frac{1}{x} + 1$. *Ans.* $x^2 + 1 + \frac{1}{x^2}$.

40. Simplify $\frac{x^2 + x(a+b) + ab}{x^2 - x(a+b) + ab} \times \frac{x^2 - a^2}{x^2 - b^2}$. *Ans.* $\frac{(x+a)^2}{(x-b)^2}$.

41. Divide $\frac{ax - x^2}{(a+x)^2}$ by $\frac{x^2}{a^2 - x^2}$. *Ans.* $\frac{(a-x)^2}{x(a+x)}$.

42. Divide $\frac{4(a^2 - ab)}{b(a+b)^2}$ by $\frac{6ab}{a^2 - b^2}$. *Ans.* $\frac{2(a-b)^2}{3b^2(a+b)}$.

43. Divide $\frac{2y^2}{x^3 + y^3}$ by $\frac{y}{y+x}$. *Ans.* $\frac{2y}{x^2 - xy + y^2}$.

44. Divide $\frac{2x+y}{x+y} + \frac{2y-x}{x-y} - \frac{x^2}{x^2-y^2}$ by $\frac{x^2+y^2}{x^2-y^2}$.
Ans. $\frac{y^2}{x^2+y^2}$.

45. Simplify $\left(\frac{x^2}{y^3} + \frac{1}{x}\right) \div \left(\frac{x}{y^2} - \frac{1}{y} + \frac{1}{x}\right)$. *Ans.* $\frac{x+y}{y}$.

46. Simplify $\left(\frac{a}{a+b} + \frac{b}{a-b}\right) \div \left(\frac{a}{a-b} - \frac{b}{a+b}\right)$. *Ans.* 1.

47. Simplify $\left(\frac{x+2y}{x+y} + \frac{x}{y}\right) \div \left(\frac{x+2y}{y} - \frac{x}{x+y}\right)$. *Ans.* 1.

48. Divide $x^4 - \frac{1}{x^4}$ by $x + \frac{1}{x}$. *Ans.* $x^3 - x + \frac{1}{x} - \frac{1}{x^3}$.

49. Divide $x^2 + \frac{1}{x^2} + 2$ by $x + \frac{1}{x}$. *Ans.* $\frac{x^2+1}{x}$.

50. Divide $x^2 + 1 + \frac{1}{x^2}$ by $\frac{1}{x} - 1 + x$. *Ans.* $\frac{x^2+x+1}{x}$.

51. Divide $a^2 - b^2 - c^2 + 2bc$ by $\frac{a+b-c}{a+b+c}$.
Ans. $a^2 - b^2 + c^2 + 2ac$.

52. Divide $\frac{a^3 + 3a^2x + 3ax^2 + x^3}{x^3 - y^3}$ by $\frac{(a+x)^2}{x^2+xy+y^2}$.
Ans. $\frac{a+x}{x-y}$.

53. Divide $a^2 - b^2 - c^2 - 2bc$ by $\dfrac{a+b+c}{a+b-c}$.

Ans. $a^2 - b^2 + c^2 - 2ac$.

54. Divide $x^2 - 3ax - 2a^2 + \dfrac{12a^3}{x+3a}$ by $3x - 6a - \dfrac{2x^2}{x+3a}$.

Ans. $\dfrac{x^2+3ax-2a^2}{x+6a}$.

55. Divide $\dfrac{x^2}{2a^2} - 4 + \dfrac{6a^2}{x^2}$ by $\dfrac{x}{2a} - \dfrac{3a}{x}$. *Ans.* $\dfrac{x^2-2a^2}{ax}$

56. Simplify $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}$. *Ans.* $\dfrac{ac-bd}{ac+bd}$.

57. Simplify $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}$. *Ans.* $\dfrac{a^2+x^2}{2ax}$.

58. Simplify $\dfrac{3abc}{bc+ca-ab} - \dfrac{\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}}{\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}}$.

Ans. $\dfrac{bc+ac+ab}{bc+ac-ab}$.

59. Simplify $\left(\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}\right) \div \left(\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}\right)$.

Ans. $-\dfrac{a^4+a^2b^2+b^4}{ab\,(a-b)^2}$.

60. Simplify $\left(\dfrac{c-b}{c+b}-\dfrac{c^3-b^3}{c^3+b^3}\right) \div \left(\dfrac{c+b}{c-b}+\dfrac{c^2+b^2}{c^2-b^2}\right)$.

Ans. $-\dfrac{bc\,(b-c)^2}{b^4+c^4+b^2c^2}$.

61. Simplify $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right) \div \left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

Ans. $\dfrac{xy}{x^2+y^2}$.

62. Simplify $\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)\div\left(\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}\right)$.

Ans. $\frac{(a^2+b^2)^2}{2a^2b^2}$.

63. Simplify $\frac{\frac{m^2+n^2}{n}-m}{\frac{1}{n}-\frac{1}{m}}\times\frac{m^2-n^2}{m^3+n^3}$. *Ans.* m.

64. Simplify $\frac{x}{x-a}-\frac{x}{x+a}-\frac{\frac{x+a}{x-a}-\frac{x-a}{x+a}}{\frac{x+a}{x-a}+\frac{x-a}{x+a}}$. *Ans.* $\frac{4a^3x}{x^4-a^4}$.

65. Simplify $\frac{\frac{1}{a}+\frac{1}{b+c}}{\frac{1}{a}-\frac{1}{b+c}}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$.

Ans. $\frac{(a+b+c)^2}{2bc}$.

66. Simplify $\frac{1}{x+\frac{1}{1+\frac{x+1}{3-x}}}$. *Ans.* $\frac{4}{3(x+1)}$.

67. Simplify $\frac{a}{b+\frac{c}{d+\frac{e}{f}}}$. *Ans.* $\frac{adf+ae}{bdf+be+cf}$.

68. Find the value of $ax+by$, when $x=\frac{cq-br}{aq-bp}$ and $y=\frac{ar-cp}{aq-bp}$. *Ans.* c.

69. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$.

Ans. 2.

70. If $\frac{a}{b}+\frac{c}{d}=1$, show that $\frac{a}{b}-\frac{c}{d}=\frac{a^2}{b^2}-\frac{c^2}{d^2}$.

71. If $\frac{a}{b}+\frac{c}{d}=\frac{a^2}{b^2}-\frac{c^2}{d^2}$, show that $\frac{a}{b}-\frac{c}{d}=1$.

175. SYNOPSIS FOR REVIEW.

CHAPTER V. FRACTIONS.

- TERMS USED . .
  - *Denominator.* } The terms of a fraction.
  - *Numerator.* }
  - *Frac. unit.*
  - *Entire quantity.*
  - *Mixed quantity.*
- SIMPLE FRAC.
  - *Proper.*
  - *Improper.*
- COMPLEX FRACTIONS.
- COMPOUND FRACTIONS.
- VALUE OF FRACTION NOT CHANGED, WHEN.
- FRACTION IN ITS LOWEST TERMS, WHEN.
- FRACTION × INTEGER. *Rule.*
- FRACTION ÷ INTEGER. *Rule.*
- TO REDUCE . . .
  - *Fraction to mixed quantity.* Rule.
  - *Entire quantity to fraction having given denominator.* Rule.
  - *Fraction to fraction having given denominator.* Rule.
  - *Mixed quantity to form of fraction.* Rule.
  - *Fractions to common denominator.* Rule.
  - *Fractions to least com. denom.* Rule.
- ADDITION. *Investigation for rule.* Rule.
- SUBTRACTION. *Investigation.* Rule.
- MULTIPLICATION. *Def. Investigation.* Rule.
- DIVISION. *Investigation.* Rule. Cor. 1, 2.
- SIGNS OF FRACTIONS.
  - *Sign prefixed to a term of numerator or denominator.*
  - *Sign prefixed to fraction.*
  - *Methods of changing sign of fraction.*
  - *Changes of sign not affecting sign of fraction.*
  - *Apparent sign of fraction.*
  - *Real sign of fraction.*

CHAPTER VI.

DEFINITIONS AND GENERAL PRINCIPLES RELATING TO EQUATIONS.

176. ***An Equation*** consists of two expressions connected by the sign of equality. Thus, $x + a = m + n$ is an equation.

The First Member of an equation is the quantity on the left of the sign of equality, and the *Second Member* is the quantity on the right of the sign. Thus, in the equation $x + a = m + n$, $x + a$ is the first member, and $m + n$ the second member.

177. ***An Identical Equation,*** or ***An Identity,*** is an equation whose members are either *identical,* or may be made identical by performing the indicated operations. Thus,

$$ax + b = ax + b, \quad \frac{a^2 - x^2}{a - x} = a + x, \quad \text{and} \quad a - \frac{a}{1 + x} = \frac{ax}{1 + x}$$

are identities.

178. It follows, from the definition, that the members of an identity are equal for *all* values that may be assigned to each letter which it contains.

Thus far the student has been almost entirely occupied with identities. Thus, the equations given in Articles **71, 72,** and **73** are identities.

179. ***An Equation of Condition*** is one whose members are equal only for a *limited* number of values of each letter which it contains. Thus, $x + 1 = 7$ is an equation of condition, because its members are not equal unless $x = 6$.

An equation of condition is called briefly, *an equation.*

180. ***An Unknown Quantity*** is a letter to which a particular value or values must be given in order that the members of an equation may become *identical.* The equation is said

to be *Satisfied* for such particular value or values. Thus, in the equation $x^2 - 4x = -3$, the unknown quantity is x, and when $x = 3$ or 1, the equation is satisfied.

181. ***An Unknown Term*** of an equation is a term containing an unknown quantity.

182. ***A Root of an Equation*** is a quantity which, when substituted for the unknown quantity, satisfies the equation. Thus, 3 and 1 are the roots of the equation $x^2 - 4x = -3$.

183. ***To solve an Equation*** is to find its roots.

184. ***A Numerical Equation*** is one in which all the *known* quantities are represented by numbers. Thus, $2x^2 + 3x = 10x + 15$ is a numerical equation.

185. ***A Literal Equation*** is one in which the known quantities are represented entirely or in part by letters. Thus, $ax + b = cx + d$ and $ax - b = 3x - 5$ are literal equations.

186. ***The Degree*** of an equation is denoted by the number of unknown factors in that term which contains the greatest number of such factors. Thus,

$ax - b = c$ is of the first degree,
$x^2 + 2px = q$ is of the second degree,
$x^2y + x^2 - cx = a$ is of the third degree,
$x^n + ax^{n-1} + bx^{n-2} = c$ is of the n^{th} degree.

REMARK.—It should be observed that the definition implies that the equation is of such a form that no unknown quantity occurs under the *radical sign*, or in a *denominator*.

187. ***A Simple Equation*** is one of the first degree.

188. ***A Quadratic Equation*** is one of the second degree.

189. ***A Cubic Equation*** is one of the third degree.

190. ***A Biquadratic Equation*** is one of the fourth degree.

191. ***Higher Equations*** are those of higher degrees than the second.

192. For brevity, the following symbols are sometimes used:

$\therefore$ signifies *hence, therefore,* or *consequently.*

$\because$ signifies *since,* or *because.*

TRANSFORMATION OF EQUATIONS.

193. ***To Transform*** an equation is to change its form without destroying the equality of its members.

194. ***Clearing of Fractions*** and ***Transposition of Terms*** are the principal transformations.

195. To clear an equation of fractions.

Let it be required to transform the equation,

$$\frac{x}{ab} - \frac{x}{bc} = d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

into another, all of whose terms shall be entire.

Multiplying both members of (1) by ab^2c, which is the product of the denominators, we obtain

$$bcx - abx = ab^2cd \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Instead of multiplying both members of (1) by ab^2c, we may clear the equation of fractions by multiplying both members by abc, which is the L. C. M. of the denominators; we thus obtain

$$cx - ax = abcd \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

RULE.

Multiply both members of the given equation by the product of all the denominators or by the L. C. M. of all the denominators.

196. To transpose a term from one member of an equation to the other.

Let us consider the equation

$$x - a = b - y \quad . \quad . \quad . \quad (1).$$

Adding a to each member of (1),

$$x - a + a = b - y + a \ (\mathbf{42},\ 2);$$

that is, $x = b + a - y$ . . . . (2).

Subtracting b from each member of (2),

$$x - b = a - y \ . \ . \ . \ . \ . \ . \ . \ (3)\ (\mathbf{42},\ 3).$$

Here we see that $-a$ has been removed from one member of the equation and appears as $+a$ in the other; and $+b$ has been removed from one member and appears as $-b$ in the other.

RULE.

Remove the term, which is to be transposed, from the member in which it stands, and write it, with its sign changed, in the other member.

COR.—If the sign of every term in an equation be changed, the equality still holds. Thus, if $x-a=b-y$, then $a-x=y-b$.

CHAPTER VII.

SIMPLE EQUATIONS.

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

197. To solve a simple equation containing only one unknown quantity.

1. Let it be required to solve the equation

$$3x - 4 = 24 - x.$$

By transposition, $\quad 3x + x = 24 + 4;$

that is, $\quad 4x = 28;$

whence, by division, $\quad x = \frac{28}{4} = 7.$

We may *verify* this result by substituting 7 for x in the given equation. The first member becomes $3 \times 7 - 4$, that is, 17; and the second member becomes $24 - 7$, that is, 17.

2. Let it be required to solve the equation

$$\frac{5x}{2} - \frac{4x}{3} - 13 = \frac{5}{8} + \frac{x}{32}.$$

Multiplying both members of this equation by 96, which is the L. C. M. of the denominators,

$$240x - 128x - 1248 = 60 + 3x.$$

By transposition,

$$240x - 128x - 3x = 1248 + 60;$$

that is, $\quad 109x = 1308;$

whence by division, $\quad x = \frac{1308}{109} = 12.$

RULE.

I. *Clear the equation of fractions, if it has any.*

II. *Transpose every unknown term of the second member to the first, and every known term of the first member to the second; and reduce each member to its simplest form.*

III. *Divide both members by the coefficient of the unknown quantity; the second member of the resulting equation will be the value of the unknown quantity.*

ILLUSTRATIONS.

Sometimes it is more convenient to clear of fractions *partially*, and then to effect some reductions before getting rid of the remaining fractional coefficients. For example,

1. Solve

$$\frac{x+7}{11}-\frac{2x-16}{3}+\frac{2x+5}{4}=5\tfrac{1}{3}+\frac{3x+7}{12}.$$

Multiplying both members by 12,

$$\frac{12(x+7)}{11}-4(2x-16)+3(2x+5)=64+3x+7;$$

that is, $\dfrac{12(x+7)}{11}-8x+64+6x+15=64+3x+7.$

By transposition and reduction,

$$\frac{12(x+7)}{11}=5x-8.$$

Multiplying by 11,

$$12x+84=55x-88;$$

by transposition, $\quad 12x-55x=-88-84;$

by reduction, $\quad -43x=-172;$

by changing signs, $\quad 43x=172;$

by division, $\quad x=\dfrac{172}{43}=4.$

2. Solve $$\frac{5}{2x+1} = \frac{2}{5x-8}.$$

Multiplying each member by $(2x+1)(5x-8)$,

$$25x - 40 = 4x + 2.$$

By transposition and reduction,

$$21x = 42;$$

by division, $$x = \frac{42}{21} = 2.$$

Verification.—Putting this value for x in the given equation, we have

$$\frac{5}{4+1} = \frac{2}{10-8}, \quad \text{or} \quad 1 = 1.$$

3. Solve $$\frac{2x-3}{3x-4} = \frac{4x-5}{6x-7}.$$

Clearing of fractions,

$$(2x-3)(6x-7) = (4x-5)(3x-4);$$

that is, $$12x^2 - 32x + 21 = 12x^2 - 31x + 20.$$

Subtracting $12x^2$ from both members,

$$21 - 32x = 20 - 31x;$$

by transposition and reduction,

$$-x = -1, \quad \text{or} \quad x = 1.$$

Verification.—Putting this value for x in the given equation, we have

$$\frac{2-3}{3-4} = \frac{4-5}{6-7}, \quad \text{or} \quad 1 = 1.$$

4. Solve $$\frac{x}{2} - 8 = \frac{10x}{3} - \frac{7}{3}.$$

Clearing of fractions,

$$3x - 48 = 20x - 14.$$

By transposition and reduction,

$$-17x = 34, \quad \text{or} \quad 17x = -34;$$

by division, $x = -\frac{34}{17} = -2.$

Verification. $\frac{-2}{2} - 8 = \frac{-20}{3} - \frac{7}{3}$, or $-9 = -9.$

5. Solve $ax + b = cx + d.$

By transposition, $ax - cx = d - b;$

that is, $(a - c)x = d - b;$

by division, $x = \frac{d-b}{a-c}.$

Verification.—Putting this value for x in the given equation, we have

$$\frac{a(d-b)}{a-c} + b = \frac{c(d-b)}{a-c} + d;$$

which is an identity, since each member reduces to $\frac{ad-bc}{a-c}.$

6. Solve $bx = a(n + x).$

By transposition, $bx - ax = an;$

by division, $x = \frac{an}{b-a}.$

Verification. $b \times \frac{an}{b-a} = a\left(n + \frac{an}{b-a}\right)$, or $\frac{abn}{b-a} = \frac{abn}{b-a}.$

198. *A simple equation containing only one unknown quantity has one root, and no more.*

By clearing of fractions, transposing, and reducing, if necessary, any simple equation containing only one unknown quantity may take the form of

$$ax = b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

whence, $x = \frac{b}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$

The value of x is verified thus:

$$a \times \frac{b}{a} = b, \quad \text{or} \quad b = b.$$

Now, it is evident that any value for x greater than $\frac{b}{a}$ would make the first member of (1) greater than the second, and that any value for x less than $\frac{b}{a}$ would make the first member of (1) less than the second. Hence, there can be no root either greater or less than $\frac{b}{a}$; that is, x is equal to $\frac{b}{a}$, and to nothing else.

This theorem may also be demonstrated thus:

Suppose the equation

$$ax = b \quad \dots\dots\dots \quad (1),$$

has two different roots, p and q; then these roots will satisfy equation (1), and give

$$ap = b \quad \dots\dots\dots \quad (2),$$

$$aq = b \quad \dots\dots\dots \quad (3).$$

Subtracting (3) from (2),

$$a(p - q) = 0 \quad \dots\dots\dots \quad (4)\ (\mathbf{42},\ 3);$$

but this is impossible, for a is not zero, and, by hypothesis, $p - q$ is not zero.

EXAMPLES.

199. Find the value of x in each of the following equations:

1. $\frac{2x+1}{2} = \frac{7x+5}{8}$. *Ans.* $x = 1$.

2. $\frac{x}{2} - 2 = \frac{x}{4} + \frac{x}{5} - 1$. *Ans.* $x = 20$.

3. $\frac{x+1}{2} + \frac{3x-4}{5} + \frac{1}{8} = \frac{6x+7}{8}$. *Ans.* $x = 3$.

4. $\frac{5x-11}{4} - \frac{x-1}{10} = \frac{11x-1}{12}$. *Ans.* $x = 11$.

5. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = \frac{1}{2}$. *Ans.* $x = \frac{6}{7}$.

6. $\frac{x+1}{2} + \frac{x+2}{3} = 16 - \frac{x+3}{4}$. *Ans.* $x = 13$.

7. $x + \dfrac{11-x}{3} = \dfrac{26-x}{2}$. *Ans.* $x = 8$.

8. $19x + \dfrac{1}{2}(7x-2) = 4x + \dfrac{35}{2}$. *Ans.* $x = 1$.

9. $\dfrac{x-3}{4} + \dfrac{x-4}{3} = \dfrac{x-5}{2} + \dfrac{x+1}{8}$. *Ans.* $x = 7$.

10. $\dfrac{5x-7}{2} - \dfrac{2x+7}{3} = 3x - 14$. *Ans.* $x = 7$.

11. $\dfrac{x-3}{4} - \dfrac{2x-5}{6} = \dfrac{41}{60} + \dfrac{3x-8}{5} - \dfrac{5x+6}{15}$. *Ans.* $x = 4$.

12. $\dfrac{5x+3}{3} - \dfrac{3x-7}{2} = 5x - 10$. *Ans.* $x = 3$.

13. $\dfrac{1}{6}(8-x) + x - 1\tfrac{2}{3} = \dfrac{x+6}{2} - \dfrac{x}{3}$. *Ans.* $x = 5$.

14. $\dfrac{x+3}{2} - \dfrac{x-2}{3} = \dfrac{3x-5}{12} + \dfrac{1}{4}$. *Ans.* $x = 28$.

15. $\dfrac{3x-1}{5} - \dfrac{13-x}{2} = \dfrac{7x}{3} - \dfrac{11(x+3)}{6}$. *Ans.* $x = 2$.

16. $\dfrac{5x-3}{7} - \dfrac{9-x}{3} = \dfrac{5x}{2} + \dfrac{19}{6}(x-4)$. *Ans.* $x = 2$.

17. $\dfrac{5x-1}{7} + \dfrac{9x-5}{11} = \dfrac{9x-7}{5}$. *Ans.* $x = 3$.

18. $\dfrac{3x+5}{7} - \dfrac{2x+7}{3} + 10 - \dfrac{3x}{5} = 0$. *Ans.* $x = 10$.

19. $\dfrac{x-1}{3} + \dfrac{4x-\frac{3}{4}}{5} - \dfrac{7x-6}{8} = 2 + \dfrac{x-2}{2} + \dfrac{3x-9}{10}$.

Ans. $x = \dfrac{4}{13}$.

20. $\left(x+\dfrac{5}{2}\right)\left(x-\dfrac{3}{2}\right) - (x+5)(x-3) + \dfrac{3}{4} = 0$.

Ans. $x = 12$.

21. $\dfrac{1}{2}\left(x-\dfrac{a}{3}\right) - \dfrac{1}{3}\left(x-\dfrac{a}{4}\right) + \dfrac{1}{4}\left(x-\dfrac{a}{5}\right) = 0$. *Ans.* $x = \dfrac{8a}{25}$.

22. $(a+x)(b+x)=(c+x)(d+x)$. *Ans.* $x=\frac{cd-ab}{a+b-c-d}$.

23. $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{b+a}$. *Ans.* $x=\frac{a^2(b-a)}{b(b+a)}$.

24. $ax+b=\frac{x}{a}+\frac{1}{b}$. *Ans.* $x=\frac{a(1-b^2)}{b(a^2-1)}$.

25. $\frac{x-a}{b}+\frac{x-b}{c}+\frac{x-c}{a}=\frac{x-(a+b+c)}{abc}$.

Ans. $x=\frac{a^2c+ab^2+bc^2-a-b-c}{ac+bc+ab-1}$.

26. $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$. *Ans.* $x=\frac{ac}{b}$.

27. $\frac{a+b}{x-c}=\frac{a}{x-a}+\frac{b}{x-b}$. *Ans.* $x=\frac{ab(a+b-2c)}{a^2+b^2-ac-bc}$.

28. $\frac{ax^2+bx+c}{px^2+qx+r}=\frac{ax+b}{px+q}$. *Ans.* $x=\frac{br-cq}{cp-ar}$.

29. $\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)b^2x}{a(a+b)^2}=3cx+\frac{bx}{a}$.

Ans. $x=\frac{ab}{a+b}$.

30. $\frac{m(x+a)}{x+b}+\frac{n(x+b)}{x+a}=m+n$. *Ans.* $x=\frac{bn-am}{m-n}$.

31. $\left(\frac{x-a}{x+b}\right)^3=\frac{x-2a-b}{x+a+2b}$. *Ans.* $x=\frac{a-b}{2}$.

32. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$.

Ans. $x=\frac{a+b+c}{3}$.

33. $.15x+1.575-.875x=.0625x$. *Ans.* $x=2$.

34. $1.2x-\frac{.18x-.05}{.5}=.4x+8.9$. *Ans.* $x=20$.

35. $4.8x-\frac{.72x-.05}{.5}=1.6x+8.9$. *Ans.* $x=5$.

SOLUTION OF PROBLEMS.

200. We shall now apply the methods already given to the solution of some problems, and thus exhibit to the student specimens of the use of Algebra.

In a problem certain quantities are given, and certain others, which have some assigned relations to them, are to be found.

201. The solution of a problem by Algebra consists of two distinct parts:

1st. *The Statement*—that is, the formation of the equation which shall express the relation between the known and the unknown quantities.

2d. *The Solution* of the equation.

202. Sometimes the conditions of the problem are such as to furnish the equation *directly;* and sometimes it is necessary, from the given conditions, to deduce others, from which to form the equation. When the conditions furnish the equation directly, they are called *Explicit Conditions.* When the conditions are deduced from those given in the problem, they are called *Implicit Conditions.*

203. It is impossible to give any precise rule for solving every problem; the following directions, however, may furnish some aid:

Denote the unknown quantity by one of the final letters of the alphabet, and express, in algebraic language, the relation which subsists between the unknown quantity and the given quantities; an equation will thus be obtained from which the value of the unknown quantity may be found.

ILLUSTRATIONS.

1. The sum of two numbers is 89, and their difference is 31; find the numbers.

Let x denote the less number, then the greater number will be $31 + x$; $\therefore$ since their sum is 89,

$$31 + x + x = 89;$$

that is,

$$31 + 2x = 89.$$

By transposition, $2x = 89 - 31 = 58$;

whence, $$x = \frac{58}{2} = 29, \text{ the less number.}$$

$\therefore$ the greater number is $29 + 31$, that is, 60.

2. A bankrupt owes B twice as much as he owes A, and C as much as he owes A and B together; out of \$300 which is to be divided among them, what should each receive?

Let x denote the number of dollars which A should receive; then, by the conditions of the problem, $2x$ will be the number of dollars B should receive, and $x + 2x$, that is, $3x$, will be the number of dollars C should receive. They together receive \$300;

$$\therefore \quad x + 2x + 3x = 300.$$

Reducing, $$6x = 300;$$

whence, $$x = \frac{300}{6} = 50;$$

therefore, A should receive \$50, B \$100, and C \$150.

3. Divide a line 21 inches long into two parts, such that one may be three-fourths of the other.

Let x denote the number of inches in one part, then $\frac{3x}{4}$ will denote the number of inches in the other part; $\therefore$ (**42**, 1),

$$x + \frac{3x}{4} = 21.$$

Clearing this equation of fractions,

$$4x + 3x = 84;$$

by reduction, $$7x = 84;$$

whence, $$x = \frac{84}{7} = 12;$$

therefore, one part is 12 inches long, and the other part 9 inches.

4. If A can perform a piece of work in 8 days, and B in 10 days, in what time will they perform it together?

Let x denote the number of days required.

In one day A can perform $\frac{1}{8}$th of the work, therefore in x days he can perform $\frac{x}{8}$ths of the work.

In one day B can perform $\frac{1}{10}$th of the work, therefore in x days he can perform $\frac{x}{10}$ths of the work. Hence, since A and B together perform the whole work in x days,

$$\frac{x}{8} + \frac{x}{10} = 1.$$

By clearing of fractions,

$$5x + 4x = 40;$$

by reduction, $$9x = 40;$$

whence, $$x = \frac{40}{9} = 4\tfrac{4}{9}.$$

5. A laborer was employed for 20 days, on condition that for every day he worked he should receive 50 cents, and for every day he was idle he should forfeit 25 cents. At the end of the 20 days he received $4; how many days did he work, and how many days was he idle?

Let x denote the number of days he worked; then he was idle $20 - x$ days.

$$50x = \text{wages due for work, and}$$

$$25\,(20 - x) = \text{the amount he forfeited;}$$

$$\therefore\ 50x - 25\,(20 - x) = 400;$$

that is, $$50x - 500 + 25x = 400;$$

by transposition and reduction,

$$75x = 900;$$

whence, $$x = 12 = \text{the number of days he worked,}$$

and $$20 - x = 8 = \text{the number of days he was idle.}$$

6. How much rye, at 54 cents a bushel, must be mixed with 50 bushels of wheat, at 72 cents a bushel, in order that the mixture may be worth 60 cents a bushel?

Let x denote the number of bushels required; then $54x$ is the value of the rye; and since the value of the wheat is 3600, the value of the mixture is $54x + 3600$.

The value of the mixture is also $(x + 50)\,60$;

$\therefore$ $$(x + 50)\,60 = 54x + 3600;$$

that is, $$60x + 3000 = 54x + 3600;$$

by transposition and reduction,

$$6x = 600;$$

whence, $$x = 100.$$

7. A smuggler had a quantity of brandy, which he expected would sell for \$100; after he had sold 10 gallons, a revenue officer seized one-third of the remainder, in consequence of which, the smuggler received only \$80 for his brandy. How many gallons had he at first, and what was the price per gallon?

Let $x =$ the number of gallons; then $\dfrac{100}{x}$ is the price per gallon, and $\dfrac{x-10}{3}$ is the quantity seized, the value of which is $100 - 80 = 20$. The value of the quantity seized is also expressed by $\dfrac{x-10}{3} \times \dfrac{100}{x}$;

$$\therefore \quad \frac{x-10}{3} \times \frac{100}{x} = 20.$$

Clearing of fractions,

$$100\,(x - 10) = 60x;$$

that is, $$100x - 1000 = 60x;$$

by transposition and reduction,

$$40x = 1000;$$

whence, $$x = 25, \text{ the number of gallons};$$

and $$\frac{100}{x} = \frac{100}{25} = 4 \text{ dollars, the price per gallon.}$$

204. *PROBLEMS.*

1. Divide \$3870 between two persons, A and B, so that A shall receive twice as much as B.

Ans. A gets \$2580, and B \$1290.

2. Divide \$420 between A and B, so that, for every dollar A receives, B may receive \$2½. *Ans.* A's share = \$120, B's = \$300.

3. How much money is there in a purse when the fourth part and the fifth part together amount to \$45? *Ans.* \$100.

4. After paying the seventh part of a bill and the fifth part, \$92 were still due; what was the amount of the bill?

Ans. \$140.

5. Divide 46 into two parts, such that if one part be divided by 7 and the other by 3, the sum of the quotients shall be 10.

Ans. 28 and 18.

6. A company of 266 persons consists of men, women, and children; there are four times as many men as children, and twice as many women as children. How many of each are there?

Ans. 38 children, 76 women, 152 men.

7. A person expends one-third of his income in board and lodging, one-eighth in clothing, one-tenth in charity, and saves \$318. What is his income? *Ans.* \$720.

8. Three towns, A, B, C, raise a sum of \$594; for every dollar B contributes, A contributes three-fifths of a dollar, and C seven-eighths of a dollar. What does each contribute?

Ans. A contributes \$144, B \$240, C \$210.

9. Divide \$1520 among A, B, and C, so that B shall have \$100 more than A, and C \$270 more than B.

Ans. A gets \$350, B \$450, C \$720.

10. A certain sum of money is to be divided among A, B, and C. A is to have \$30 less than the half, B is to have \$10 less than the third part, and C is to have \$8 more than the fourth part. What does each receive? *Ans.* A \$162, B \$118, C \$104.

11. The sum of two numbers is 5760, and their difference is equal to one-third of the greater; find the numbers.

Ans. 3456 and 2304.

12. Two casks contain equal quantities of beer; from the first 34 quarts are drawn, and from the second 80; the quantity remaining in the first cask is now twice that in the second. How much did each cask originally contain? *Ans.* 126 quarts.

13. A person bought a picture at a certain price, and paid the same price for a frame; if the frame had cost \$1 less, and the picture \$$\frac{3}{4}$ more, the price of the frame would have been only half that of the picture. What was the price of the picture?

Ans. \$$2\frac{3}{4}$.

14. A house and garden cost \$850, and five times the price of the house was equal to twelve times the price of the garden; find the price of each. *Ans.* House, \$600; garden, \$250.

15. One-tenth of a rod is colored red, one-twentieth orange, one-thirtieth yellow, one-fortieth green, one-fiftieth blue, one-sixtieth indigo, and the remainder, which is 302 inches long, violet; find the length of the rod. *Ans.* 400 inches.

16. Two-thirds of a certain number of persons received \$18 each, and one-third received \$30 each. The whole sum received was \$660. How many persons were there? *Ans.* 30.

17. Find the number whose third part added to its seventh part gives a sum equal to 20. *Ans.* 42.

18. The difference between the squares of two consecutive numbers is 15. What are the numbers? *Ans.* 7 and 8.

19. A performs $\frac{2}{7}$ of a piece of work in 4 days; he then receives the assistance of B, and the two together finish it in 6 days. Find the time in which each alone can do the whole work.

Ans. A in 14 days; B in 21 days.

20. A bought eggs at 18 cents a dozen; had he bought 5 more for the same money, they would have cost him $2\frac{1}{2}$ cents a dozen less. How many eggs did he buy? *Ans.* 31.

21. A man bought a certain number of sheep for \$94; having lost 7 of them, he sold one-fourth of the remainder at prime cost for \$20. How many sheep did he buy? *Ans.* 47.

22. A man leaves home in a stage which travels 12 miles an hour, and agrees to return in 2 hours. How far may he ride if he walks back at the rate of 4 miles an hour? *Ans.* 6 miles.

23. A and B play at a game, agreeing that the loser shall always pay to the winner \$1 more than half the money the loser has; they commence with equal sums of money, but after B has lost the first game and won the second, he has twice as much as A; how much had each at the beginning? *Ans.* \$6.

24. A person who possesses \$12000 uses a portion of the money in building a house. One-third of the money which remains he invests at 4 per cent., and the other two-thirds at 5 per cent., and from these investments he obtains an income of \$392. What was the cost of the house? *Ans.* \$3600.

25. A takes from a purse \$2 and one-sixth of what remains; then B takes \$3 and one-sixth of what remains; they then find that they have taken equal amounts. How many dollars were in the purse, and how many did each take?

Ans. There were \$20 in the purse, and each took \$5.

26. A vessel can be emptied by three taps; by the first alone it could be emptied in 80 minutes; by the second alone, in 200 minutes; and by the third alone, in 5 hours. In what time will the vessel be emptied if all the taps are opened? *Ans.* 48 min.

27. A person buys some tea at 36 cents a pound, and some at 60 cents a pound; he wishes to mix them, so that, by selling the mixture at 44 cents a pound, he may gain 10 per cent. on each pound sold; find how many pounds of the inferior tea he must mix with each pound of the superior. *Ans.* 5.

28. A cask, A, contains 12 gallons of wine and 18 gallons of water; another cask, B, contains 9 gallons of wine and 3 gallons of water; how many gallons must be drawn from each cask, so as to produce, by their mixture, 7 gallons of wine and 7 gallons of water? *Ans.* 10 from A, and 4 from B.

29. A can dig a ditch in one-half the time that B can; B can dig it in two-thirds of the time that C can; all together they can dig it in 6 days; find the time in which each alone can dig the ditch. *Ans.* A in 11 days, B in 22 days, and C in 33 days.

30. At what time between one o'clock and two o'clock is the minute hand exactly one minute in advance of the hour hand?

Ans. $6\frac{6}{11}$ minutes past one.

31. A man leaves home in a stage which travels b miles an hour, and agrees to return in a hours. How far may he ride, if he walks back at the rate of c miles an hour?

Ans. $\frac{abc}{b+c}$ miles.

SCH.—As a, b, and c may have any values whatever, the solution of Problem 31 furnishes a formula which can be used for the solution of any similar problem. Thus, to obtain the answer to Problem 22, we have only to substitute 2 for a, 12 for b, and 4 for c, which gives

$$x = \frac{2 \times 12 \times 4}{12 + 4} = \frac{96}{16} = 6.$$

A problem is said to be *generalized* when letters are nsed to represent its known quantities.

32. A crew, which can row a boat at the rate of 9 miles an hour in still water, finds that it takes twice as long to come up a river as to go down; at what rate does the river flow?

Ans. 3 miles an hour.

33. A certain article of consumption is subject to a duty of 72 cents per cwt.; in consequence of a reduction in the duty, the consumption increases one-half, but the revenue falls one-third. Find the duty per cwt. after the reduction.

Ans. 32 cents per cwt.

34. A merchant maintained himself for 3 years at a cost of \$250 a year; and in each of those years augmented that part of his stock which was not so expended by one-third thereof. At the end of the third year his original stock was doubled; what was that stock? *Ans.* \$3700.

35. A market woman bought some eggs at 2 for a cent, and as many more at 3 for a cent; she sold them all at the rate of 5 for 2 cents, and found she had lost 4 cents. How many did she buy of each sort? *Ans.* 120.

36. A man hired a servant for one year at the wages of \$90 and a suit of clothes. At the end of 7 months the servant quits his service and receives \$33.75 and the suit of clothes. At what price were the clothes estimated? *Ans.* \$45.

37. A general arranging his men in the form of a solid square, finds he has 21 men over; but attempting to add one man to each side of the square, finds he wants 200 men to fill up the square; find the number of men. *Ans.* 12121.

38. A boatman can row 14 miles an hour with the tide; *against* a tide two-thirds as strong, he can row only four miles an hour. What is the velocity of the tide in each case?

Ans. 6 miles, and 4 miles.

39. Two men start from the same point at the same time, and travel in the same direction; the first steps twice as far as the second, but the second makes five steps while the first makes one. At the eud of a certain time they are 300 feet apart; how far has each traveled? *Ans.* 1st, 200 feet; 2d, 500 feet.

40. A ship sails with a supply of biscuit for 60 days, at a daily allowance of a pound a head; after being at sea 20 days she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's daily allowance must be reduced to five-sevenths of a pound. Find the original number of the crew. *Ans.* 40.

SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES.

205. Suppose we have an equation containing two unknown quantities, x and y; for example,

$$5x - 2y = 4 \quad . \quad . \quad . \quad . \quad (1).$$

For every value which we please to ascribe to one of the unknown quantities we can determine the corresponding value of the other, and thus find as many pairs of values as we please. Thus, if $y = 1$, we find $x = \frac{6}{5}$; if $y = 2$, we find $x = \frac{8}{5}$; and so on.

Suppose we have another equation of the same kind; for example,

$$4x + 3y = 17 \quad . \quad . \quad . \quad (2).$$

We can also find as many pairs of values as we please which satisfy this equation.

But suppose we ask for values of x and y which satisfy *both* equations; we shall find then that there is only one value of x and one value of y. For, multiplying (1) by 3, and (2) by 2, we have

$$15x - 6y = 12 \quad . \quad . \quad . \quad (3),$$

$$8x + 6y = 34 \quad . \quad . \quad . \quad (4).$$

Adding (3) and (4), member to member (**42**, 2),

$$23x = 46 \quad . \quad . \quad . \quad . \quad . \quad (5);$$

whence, $x = 2$.

We may now find the value of y by substituting 2 for x in either of the given equations. Substituting 2 for x in (1),

$$10 - 2y = 4 \quad . \quad . \quad . \quad . \quad (6);$$

whence, $y = 3$.

Hence, if *both* equations are to be satisfied, x must be equal to 2, and y must be equal to 3.

206. ***Simultaneous Equations*** are two or more equations which are to be satisfied by the *same values* of the unknown quantities. We are now about to treat of simultaneous equations of the first degree involving *two* unknown quantities.

207. There are three methods which are usually given for solving these equations. The object of each method is to obtain from the *two* given equations a single equation containing only *one* of the unknown quantities. The unknown quantity which does not appear in the resulting equation is said to be *eliminated*.

208. ELIMINATION BY ADDITION OR SUBTRACTION.

1. Let it be required to solve the equations,

$$4x + 3y = 22 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$5x - 7y = 6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Multiplying (1) by 7, and (2) by 3,

$$28x + 21y = 154 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$15x - 21y = 18 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Adding (3) and (4),

$$43x = 172 \quad . \quad . \quad . \quad . \quad . \quad (5);$$

whence, $x = 4.$

Substituting 4 for x in (1),

$$16 + 3y = 22;$$

whence, $y = 2.$

2. Let it be required to solve the equations,

$$2x + 7y = 29. \quad . \quad . \quad . \quad (1),$$

$$3x + 5y = 27 \quad . \quad . \quad . \quad (2).$$

Multiplying (1) by 5, and (2) by 7,

$$10x + 35y = 145 \quad . \quad . \quad (3),$$

$$21x + 35y = 189 \quad . \quad . \quad (4).$$

Subtracting (3) from (4),

$$11x = 44 \quad . \quad . \quad . \quad . \quad . \quad (5);$$

whence, $x = 4.$

Substituting 4 for x in (1),

$$8 + 7y = 29;$$

whence, $y = 3.$

RULE.

I. *Multiply or divide the given equations by such quantities that the coefficients of the quantity to be eliminated shall be equal in the two resulting equations.*

II. *If these equal coefficients have like signs, subtract one of the resulting equations from the other, member from member; if they have unlike signs, add the equations, member to member.*

Sch. 1.—Before commencing the operation of elimination, each of the given equations should be reduced to the form of $ax + by = c$, if it is not already of that form.

SCH. 2.—The coefficients of the quantity to be eliminated *may* be equal in the given equations; in that case, the first step in the rule is unnecessary.

SCH. 3.—In preparing the given equations by multiplication, it is best to divide the L. C. M. of the coefficients of the quantity to be eliminated by each of these coefficients; the quotients thus obtained will be the *least multipliers* that can be used.

SCH. 4.—It is generally convenient to clear the equations of fractions, if they have any, before applying the rule. This is not necessary, however. For if the quantity to be eliminated has fractional coefficients in the two equations, they may be reduced to equivalent fractions having a common denominator; it will then be necessary to render the numerators equal by multiplication or division, according to the rule.

EXAMPLES.

Solve the following groups of simultaneous equations:

1. $\left\{\begin{array}{l}5x+6y=49\\7x-4y=19\end{array}\right\}.$ *Ans.* $x=5,\ y=4.$

2. $\left\{\begin{array}{l}6x+5y=61\\3x+4y=38\end{array}\right\}.$ *Ans.* $x=6,\ y=5.$

3. $\left\{\begin{array}{r}x+3y=10\\3x+2y=9\end{array}\right\}.$ *Ans.* $x=1,\ y=3.$

4. $\left\{\begin{array}{r}x-\frac{1}{4}(y-2)=5\\4y-\frac{1}{3}(x+10)=3\end{array}\right\}.$ *Ans.* $x=5,\ y=2.$

5. $\left\{\begin{array}{l}x+y=s\\x-y=d\end{array}\right\}.$ *Ans.* $x=\dfrac{s+d}{2},\ y=\dfrac{s-d}{2}.$

209. ELIMINATION BY SUBSTITUTION.

Let it be required to solve the equations

$$4x+3y=22 \quad . \quad . \quad . \quad (1),$$

$$5x-7y=6 \quad . \quad . \quad . \quad . \quad (2).$$

From (1) we find

$$y=\frac{22-4x}{3} \quad . \quad . \quad . \quad . \quad (3).$$

Substituting this value for y in (2),

$$5x - 7 \times \frac{22 - 4x}{3} = 6;$$

that is, $$5x - \frac{154 - 28x}{3} = 6 \quad . \quad . \quad . \quad (4).$$

Multiplying (4) by 3,

$$15x - 154 + 28x = 18 \quad . \quad . \quad (5);$$

by transposition and reduction,

$$43x = 172;$$

whence, $$x = 4.$$

Substituting 4 for x in (3), we find $y = 2$.

RULE.

Find, from one of the given equations, an expression for the value of the unknown quantity to be eliminated, and substitute this value for the same unknown quantity in the other equation; there will thus be formed a new equation containing only one unknown quantity.

EXAMPLES.

Solve the following groups of simultaneous equations:

1. $\left\{\begin{array}{l} 2x + 3y = 33 \\ 4x + 5y + 59 \end{array}\right\}.$ *Ans.* $x = 6, \ y = 7.$

2. $\left\{\begin{array}{l} \dfrac{x+y}{2} - \dfrac{x-y}{3} = 8 \\ \dfrac{x+y}{3} + \dfrac{x-y}{4} = 11 \end{array}\right\}.$ *Ans.* $x = 18, \ y = 6.$

3. $\left\{\begin{array}{l} 3x - 2y = 1 \\ 3y - 4x = 1 \end{array}\right\}.$ *Ans.* $x = 5, \ y = 7.$

4. $\left\{\begin{array}{l} \dfrac{x}{2} + \dfrac{y}{3} = 1 \\ \dfrac{x}{3} + \dfrac{y}{4} = 1 \end{array}\right\}.$ *Ans.* $x = -6, \ y = 12.$

5. $\left.\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = m \\ \dfrac{x}{c} + \dfrac{y}{d} = n \end{cases}\right\}$. *Ans.* $x = \dfrac{ac(dn - bm)}{ad - bc}$, $y = \dfrac{bd(am - cn)}{ad - bc}$.

210. ELIMINATION BY COMPARISON.

Let it be required to solve the equations

$$7x + 6y = 20 \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$9x - 4y = 14 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

From (1), $y = \dfrac{20 - 7x}{6}$ . . . . . . (3),

and from (2), $y = \dfrac{9x - 14}{4}$ . . . . . . (4);

∴ (42, 6), $\dfrac{9x - 14}{4} = \dfrac{20 - 7x}{6}$ . . . (5).

Clearing of fractions,

$$27x - 42 = 40 - 14x;$$

whence, $x = 2$.

Substituting 2 for x in either (3) or (4), we find $y = 1$.

RULE.

Find, from each of the given equations, an expression for the value of the unknown quantity to be eliminated, and equate the expressions thus obtained; an equation will thus be formed containing only one unknown quantity.

EXAMPLES.

Solve the following groups of simultaneous equations:

1. $\left.\begin{cases} 4x - 2y = 20 \\ 4x + 2y = 100 \end{cases}\right\}$. *Ans.* $x = 15$, $y = 20$.

2. $\left\{\begin{array}{l} 3x + 4y = 18 \\ 2x - y = 1 \end{array}\right\}$. *Ans.* $x = 2,\ y = 3$.

3. $\left\{\begin{array}{l} 7x - 3y = 12 \\ 2x + 2y = 12 \end{array}\right\}$. *Ans.* $x = 3,\ y = 3$.

4. $\left\{\begin{array}{l} 2x - \frac{3}{4}y = 9 \\ x + \phantom{\frac{3}{4}}y = 21 \end{array}\right\}$. *Ans.* $x = 9,\ y = 12$.

5. $\left\{\begin{array}{l} \dfrac{2x - y}{4} - \dfrac{3}{2} = \dfrac{3y}{4} - x - 2 \\ \dfrac{x + y}{3} = 2\frac{2}{3} \end{array}\right\}$. *Ans.* $x = 3,\ y = 5$.

211. General Scholium.—In the solution of simultaneous equations, any of the preceding methods of elimination can be used, as may be most convenient, each method having its advantages in particular cases. Generally, however, the equation obtained by using the second or third method contains fractional terms. This inconvenience is avoided if we eliminate by the first method. The second method may be preferable whenever the coefficient of one of the unknown quantities in one of the given equations is unity; for then the inconvenience of which we have just spoken may be avoided. We shall sometimes have occasion to use the second and third methods, but generally the first method is preferable.

EXAMPLES.

Solve the following groups of simultaneous equations:

1. $\left\{\begin{array}{l} x + y = 15 \\ x - y = 7 \end{array}\right\}$. *Ans.* $x = 11,\ y = 4$.

2. $\left\{\begin{array}{l} 3x - 2y = 1 \\ 3y - 4x = 1 \end{array}\right\}$. *Ans.* $x = 5,\ y = 7$.

3. $\left\{\begin{array}{l} 3x - 5y = 13 \\ 2x + 7y = 81 \end{array}\right\}$. *Ans.* $x = 16,\ y = 7$.

4. $\left\{\begin{array}{l} 2x + 3y = 43 \\ 10x - y = 7 \end{array}\right\}$. *Ans.* $x = 2,\ y = 13$.

5. $\left\{\begin{array}{l} 5x - 7y = 33 \\ 11x + 12y = 100 \end{array}\right\}.$ *Ans.* $x = 8,\ y = 1.$

6. $\left\{\begin{array}{l} 3y - 7x = 4 \\ 2y + 5x = 22 \end{array}\right\}.$ *Ans.* $x = 2,\ y = 6.$

7. $\left\{\begin{array}{l} 21y + 20x = 165 \\ 77y - 30x = 295 \end{array}\right\}.$ *Ans.* $x = 3,\ y = 5.$

8. $\left\{\begin{array}{l} 5x + 7y = 43 \\ 11x + 9y = 69 \end{array}\right\}.$ *Ans.* $x = 3,\ y = 4.$

9. $\left\{\begin{array}{l} 8x - 21y = 33 \\ 6x + 35y = 177 \end{array}\right\}.$ *Ans.* $x = 12,\ y = 3.$

10. $\left\{\begin{array}{l} 11x - 10y = 14 \\ 5x + 7y = 41 \end{array}\right\}.$ *Ans.* $x = 4,\ y = 3.$

11. $\left\{\begin{array}{l} 16x + 17y = 500 \\ 17x - 3y = 110 \end{array}\right\}.$ *Ans.* $x = 10,\ y = 20.$

12. $\left\{\begin{array}{l} \frac{x}{5} + \frac{y}{6} = 18 \\ \frac{x}{2} - \frac{y}{4} = 21 \end{array}\right\}.$ *Ans.* $x = 60,\ y = 36.$

13. $\left\{\begin{array}{l} \frac{x}{3} + \frac{y}{4} = 9 \\ \frac{x}{4} + \frac{y}{5} = 7 \end{array}\right\}.$ *Ans.* $x = 12,\ y = 20.$

14. $\left\{\begin{array}{l} \frac{x}{2} - \frac{y}{3} = -7 \\ \frac{x}{3} + \frac{y}{4} = 1 \end{array}\right\}.$ *Ans.* $x = -6,\ y = 12.$

15. $\left\{\begin{array}{l} \frac{x+y}{2} - \frac{x-y}{3} = 8 \\ \frac{x+y}{3} - \frac{x-y}{4} = 5 \end{array}\right\}.$ *Ans.* $x = 18,\ y = 6.$

16. $\left\{\begin{array}{l}\dfrac{11x-5y}{11}=\dfrac{3x+y}{16} \\ 8x-5y=1\end{array}\right\}.$ *Ans.* $x=7,\ y=11.$

17. $\left\{\begin{array}{l}\dfrac{2x}{3}-4+\dfrac{y}{2}+x=8-\dfrac{3y}{4}+\dfrac{1}{12} \\ \dfrac{y}{6}-\dfrac{x}{2}+2=\dfrac{1}{6}-2x+6\end{array}\right\}.$ *Ans.* $x=2,\ y=7.$

18. $\left\{\begin{array}{l}x=4y \\ \dfrac{1}{5}(2x+7y)-1=\dfrac{2}{3}(2x-6y+1)\end{array}\right\}.$ *Ans.* $x=4,\ y=1.$

19. $\left\{\begin{array}{l}x+\dfrac{1}{2}(3x-y-1)=\dfrac{1}{4}+\dfrac{3}{4}(y-1) \\ \dfrac{1}{5}(4x+3y)=\dfrac{7y}{10}+2\end{array}\right\}.$

Ans. $x=3\frac{1}{3},\ y=6\frac{2}{3}.$

20. $\left\{\begin{array}{l}\dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5} \\ 8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3}\end{array}\right\}.$ *Ans.* $x=12,\ y=6.$

21. $\left\{\begin{array}{l}\dfrac{3x}{10}-\dfrac{y}{15}-\dfrac{4}{9}=\dfrac{x}{12}-\dfrac{y}{18} \\ 2x-\dfrac{8}{3}=\dfrac{x}{12}-\dfrac{y}{15}+\dfrac{11}{10}\end{array}\right\}.$ *Ans.* $x=2,\ y=-1.$

22. $\left\{\begin{array}{l}\dfrac{4x-3y-7}{5}=\dfrac{3x}{10}-\dfrac{2y}{15}-\dfrac{5}{6} \\ \dfrac{y-1}{3}+\dfrac{x}{2}-\dfrac{3y}{20}=\dfrac{y-x}{15}+\dfrac{x}{6}+\dfrac{11}{10}\end{array}\right\}.$ *Ans.* $x=3,\ y=2.$

23. $\left\{\begin{array}{l}\dfrac{\dfrac{2x}{3}-\dfrac{5y}{12}}{\dfrac{7}{4}}-\dfrac{\dfrac{3x}{2}-\dfrac{y}{3}}{\dfrac{23}{2}}=2 \\ \dfrac{x-y}{x+y}=\dfrac{1}{5}\end{array}\right\}.$ *Ans.* $x=18,\ y=12.$

24. $\left\{\begin{aligned} 13x + 11y &= 4a \\ 12x - 6y &= a \end{aligned}\right\}$. *Ans.* $x = y = \dfrac{a}{6}$.

25. $\left\{\begin{aligned} \frac{x}{m} + \frac{y}{n} &= 2 \\ \frac{x}{m} - \frac{y}{n} &= 1 \end{aligned}\right\}$. *Ans.* $x = \dfrac{3m}{2},\ y = \dfrac{n}{2}$.

26. $\left\{\begin{aligned} \frac{x}{a} - \frac{y}{b} &= 1 \\ \frac{x}{3a} - \frac{y}{6b} &= \frac{2}{3} \end{aligned}\right\}$. *Ans.* $x = 3a,\ y = 2b$.

27. $\left\{\begin{aligned} ax + by &= c \\ mx - ny &= d \end{aligned}\right\}$. *Ans.* $x = \dfrac{cn + bd}{bm + an},\ y = \dfrac{cm - ad}{bm + an}$.

28. $\left\{\begin{aligned} \frac{x}{b+c} + \frac{y}{a+c} &= 2 \\ \frac{ax - by}{(a-b)c} &= 1 \end{aligned}\right\}$. *Ans.* $x = b + c,\ y = a + c$.

29. $\left\{\begin{aligned} \frac{x}{a+b} + \frac{y}{a-b} &= 2a \\ \frac{x-y}{4ab} &= 1 \end{aligned}\right\}$. *Ans.* $x = (a + b)^2,\ y = (a - b)^2$.

SIMPLE EQUATIONS WITH ANY NUMBER OF UNKNOWN QUANTITIES.

212. To solve a group of simple equations containing any number of unknown quantities.

Let it be required to solve the equations

$$3x + 4y - 2z = 10 \quad . \quad . \quad . \quad (1),$$

$$5x - 2y + 3z = 16 \quad . \quad . \quad . \quad (2),$$

$$4x + 2y + 2z = 22 \quad . \quad . \quad . \quad (3).$$

Combining (1) and (2), also (1) and (3), eliminating z in each case, we have the new group

$$19x + 8y = 62 \quad . \quad . \quad . \quad (4),$$
$$7x + 6y = 32 \quad . \quad . \quad . \quad (5).$$

Combining (4) and (5), eliminating y, we have

$$29x = 58;$$

whence, $$x = 2.$$

Substituting 2 for x in (5), we have

$$14 + 6y = 32;$$

whence, $$y = 3.$$

Substituting 2 for x and 3 for y in (1), we have

$$6 + 12 - 2z = 10;$$

whence, $$z = 4.$$

RULE.

I. *Combine one equation of the group with each of the others, eliminating the same unknown quantity in each case; there will result a new group containing one equation less than the original group.*

II. *Combine one equation of the resulting group with each of the others, eliminating a second unknown quantity; there will result a new group containing two equations less than the original group.*

III. *Continue the operation until a single equation is found, containing only one unknown quantity.*

IV. *Find the value of this unknown quantity by the rule of Art.* **197**; *substitute this value in either one of the group of two equations, and find the value of a second unknown quantity; then substitute the two values thus found in any one of the group of three equations, and find the value of a third unknown quantity; and so on, till the values of all are found.*

SCH.—When any one of the unknown quantities does not occur in all the equations, it will generally be best to eliminate that quantity first.

EXAMPLES.

Solve the following groups of simultaneous equations:

1. $\left\{\begin{array}{r} x - 2y + 3z = 2 \\ 2x - 3y + z = 1 \\ 3x - y + 2z = 9 \end{array}\right\}$. *Ans.* $x = 3$, $y = 2$, $z = 1$.

2. $\left\{\begin{array}{r} 3x + 2y - 4z = 15 \\ 5x - 3y + 2z = 28 \\ 3y + 4z - x = 24 \end{array}\right\}$. *Ans.* $x = 7$, $y = 5$, $z = 4$.

3. $\left\{\begin{array}{r} x + y - z = 1 \\ 8x + 3y - 6z = 1 \\ 3z - 4x - y = 1 \end{array}\right\}$. *Ans.* $x = 2$, $y = 3$, $z = 4$.

4. $\left\{\begin{array}{r} 2x - 7y + 4z = 0 \\ 3x - 3y + z = 0 \\ 9x + 5y + 3z = 28 \end{array}\right\}$. *Ans.* $x = 1$, $y = 2$, $z = 3$.

5. $\left\{\begin{array}{r} 4x - 3y + 2z = 9 \\ 2x + 5y - 3z = 4 \\ 5x + 6y - 2z = 18 \end{array}\right\}$. *Ans.* $x = 2$, $y = 3$, $z = 5$.

6. $\left\{\begin{array}{r} 2x - 4y + 9z = 28 \\ 7x + 3y - 5z = 3 \\ 9x + 10y - 11z = 4 \end{array}\right\}$. *Ans.* $x = 2$, $y = 3$, $z = 4$.

7. $\left\{\begin{array}{r} x - 2y + 3z = 6 \\ 2x + 3y - 4z = 20 \\ 3x - 2y + 5z = 26 \end{array}\right\}$. *Ans.* $x = 8$, $y = 4$, $z = 2$.

8. $\left\{\begin{array}{r} 4x - 3y + 2z = 40 \\ 5x + 9y - 7z = 47 \\ 9x + 8y - 3z = 97 \end{array}\right\}$. *Ans.* $x = 10$, $y = 2$, $z = 3$.

9. $\left\{\begin{array}{r} 3x + 2y + z = 23 \\ 5x + 2y + 4z = 46 \\ 10x + 5y + 4z = 75 \end{array}\right\}$. *Ans.* $x = 4$, $y = 3$, $z = 5$.

10. $\left.\begin{cases} 5x - 6y + 4z = 15 \\ 7x + 4y - 3z = 19 \\ 2x + \ \ y + 6z = 46 \end{cases}\right\}.$ *Ans.* $x = 3$, $y = 4$, $z = 6$.

11. $\left.\begin{cases} x + y + z = 31 \\ x + y - z = 25 \\ x - y - z = \ \ 9 \end{cases}\right\}.$ *Ans.* $x = 20$, $y = 8$, $z = 3$.

12. $\left.\begin{cases} x + y + z = 26 \\ x - y \qquad = \ \ 4 \\ x - z \qquad = \ \ 6 \end{cases}\right\}.$ *Ans.* $x = 12$, $y = 8$, $z = 6$.

13. $\left.\begin{cases} x - y - z = \ \ 6 \\ 3y - x - z = 12 \\ 7z - y - x = 24 \end{cases}\right\}.$ *Ans.* $x = 39$, $y = 21$, $z = 12$.

14. $\left.\begin{cases} \dfrac{3y-1}{4} = \dfrac{6z}{5} - \dfrac{x}{2} + \dfrac{9}{5} \\ \dfrac{5x}{4} + \dfrac{4z}{3} = y + \dfrac{5}{6} \\ \dfrac{3x+1}{7} - \dfrac{z}{14} + \dfrac{1}{6} = \dfrac{2z}{21} + \dfrac{y}{3} \end{cases}\right\}.$ *Ans.* $x=2$, $y=3$, $z=1$.

15. $\left.\begin{cases} \dfrac{10x + 4y - 5z}{5} = \dfrac{4x + 6y - 3z}{9} \\ 10x + 4y - 5z = 4x + 6y - 3z - 8 \\ \dfrac{10x + 4y - 5z}{10} + \dfrac{4x + 6y - 3z}{3} = \dfrac{x + y + z}{4} \end{cases}\right\}.$

Ans. $x = 6$, $y = \dfrac{20}{3}$, $z = \dfrac{46}{3}$.

16. $\left.\begin{cases} 7x - 3y = 1 \\ 11z - 7u = 1 \\ 4z - 7y = 1 \\ 19x - 3u = 1 \end{cases}\right\}.$ *Ans.* $x = 4$, $y = 9$, $z = 16$, $u = 25$.

17. $\left.\begin{cases} 3u - 2y = \ \ 2 \\ 5x - 7z = 11 \\ 2x + 3y = 39 \\ 4y + 3z = 41 \end{cases}\right\}.$ *Ans.* $x = 12$, $y = 5$, $z = 7$, $u = 4$.

18. $\left.\begin{cases} 2x - 3y + 2z = 13 \\ \qquad 4y + 2z = 14 \\ \qquad 4u - 2x = 30 \\ \qquad 5y + 3u = 32 \end{cases}\right\}$. *Ans.* $x=3$, $y=1$, $z=5$, $u=9$.

19. $\left.\begin{cases} 7u - 13z = 87 \\ 10y - 3x = 11 \\ 3u + 14x = 57 \\ 2x - 11z = 50 \end{cases}\right\}$. *Ans.* $x = 3$, $y = 2$, $z = -4$, $u = 5$.

20. $\left.\begin{cases} 7x - 2z + 3u = 17 \\ 4y - 2z + v = 11 \\ 5y - 3x - 2u = 8 \\ 4y - 3u + 2v = 9 \\ 3z + 8u = 33 \end{cases}\right\}$.

Ans. $x = 2$, $y = 4$, $z = 3$, $u = 3$, $v = 1$.

21. $\left.\begin{cases} 3x - 4y + 3z + 3v - 6u = 11 \\ \qquad 3x - 5y + 2z - 4u = 11 \\ \qquad 10y - 3z + 3u - 2v = 2 \\ \qquad 5z + 4u + 2v - 2x = 3 \\ \qquad 6u - 3v + 4x - 2y = 6 \end{cases}\right\}$.

Ans. $x = 2$, $y = 1$, $z = 3$, $u = -1$, $v = -2$.

22. $\left.\begin{cases} \frac{x}{a} + \frac{y}{b} = 1 \\ \frac{x}{a} + \frac{z}{c} = 1 \\ \frac{y}{b} + \frac{z}{c} = 1 \end{cases}\right\}$. *Ans.* $x = \frac{a}{2}$, $y = \frac{b}{2}$, $z = \frac{c}{2}$.

23. $\left.\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases}\right\}$.

Ans. $x = \dfrac{b^2 + c^2 - a^2}{2bc}$, $y = \dfrac{a^2 + c^2 - b^2}{2ac}$, $z = \dfrac{a^2 + b^2 - c^2}{2ab}$.

24. $\left.\begin{cases} x + a = y + z \\ y + a = 2x + 2z \\ z + a = 3x + 3y \end{cases}\right\}$. *Ans.* $x = \frac{1}{11}a$, $y = \frac{5}{11}a$, $z = \frac{7}{11}a$.

25. $\left\{\begin{array}{l} x+y+z=0 \\ (b+c)x+(a+c)y+(a+b)z=0 \\ bcx+acy+abz=1 \end{array}\right\}.$

Ans. $x=\dfrac{1}{(a-b)(a-c)}$, $y=\dfrac{1}{(b-a)(b-c)}$, $z=\dfrac{1}{(c-a)(c-b)}$.

26. $\left\{\begin{array}{l} ax+by+cz=\mathrm{A} \\ a^2x+b^2y+c^2z=\mathrm{A}^2 \\ a^3x+b^3y+c^3z=\mathrm{A}^3 \end{array}\right\}.$ *Ans.* $x=\dfrac{\mathrm{A}(\mathrm{A}-b)(\mathrm{A}-c)}{a(a-b)(a-c)}$.

27. $\left\{\begin{array}{l} x+y+z=a+b+c \\ bx+cy+az=cx+ay+bz \\ cx+ay+bz=a^2+b^2+c^2 \end{array}\right\}.$ *Ans.* $x=b+c-a$.

28. $\left\{\begin{array}{l} x-ay+a^2z=a^3 \\ x-by+b^2z=b^3 \\ x-cy+c^2z=c^3 \end{array}\right\}.$ *Ans.* $x=abc$.

29. $\left\{\begin{array}{l} cx+y+az=2a \\ c^2x+y+a^2z=2ac \\ acx-y+acz=a^2+c^2 \end{array}\right\}.$

Ans. $x=\dfrac{a+1}{c}$, $y=a-c$, $z=\dfrac{c-1}{a}$.

30. $\left\{\begin{array}{l} u+v+w+x+y=15 \\ v+w+x+y+z=20 \\ w+x+y+z+u=19 \\ x+y+z+u+v=18 \\ y+z+u+v+w=17 \\ z+u+v+w+x=16 \end{array}\right\}.$

Ans. $u=1$, $v=2$, $w=3$, $x=4$, $y=5$, $z=6$.

213. *PROBLEMS.*

1. A and B engage in play; in the first game A wins as much as he had and four dollars more, and finds he has twice as much as B; in the second game B wins half as much as he had at first and one dollar more, and then it appears he has three times as much as A; what sum had each at first?

Let $x=$ the number of dollars which A had, and
$y=$ the number of dollars which B had;

then after the first game A has $2x+4$ dollars, and B has $y-x-4$ dollars.

$\therefore$ by the first condition,

$$2x + 4 = 2(y - x - 4) \quad . \quad . \quad . \quad (1).$$

Again, after the second game A has $2x + 4 - \frac{y}{2} - 1$ dollars, and B has $y - x - 4 + \frac{y}{2} + 1$ dollars.

$\therefore$ by the second condition,

$$y - x - 4 + \frac{y}{2} + 1 = 3\left(2x + 4 - \frac{y}{2} - 1\right) \quad . \quad . \quad . (2).$$

By transposition and reduction, (1) and (2) become,

$$y - 2x = 6 \quad . \quad . \quad . \quad (3),$$

$$3y - 7x = 12 \quad . \quad . \quad . \quad (4).$$

Multiplying (3) by 3,

$$3y - 6x = 18 \quad . \quad . \quad . \quad (5).$$

Subtracting (4) from (5),

$$x = 6.$$

Substituting 6 for x in (3), we find

$$y = 18.$$

2. A sum of money was divided equally among a certain number of persons; had there been three more persons, each would have received one dollar less, and had the number of persons been two less, each would have received one dollar more than he did; what was the number of persons, and what did each receive?

Let $x =$ the number of persons, and

$y =$ the number of dollars each received;

then xy dollars is the sum divided.

By the conditions of the problem, the sum divided is also expressed by $(x + 3)(y - 1)$, or $(x - 2)(y + 1)$; $\therefore$ (**42**, 6), we have,

$$(x+3)(y-1) = xy \quad . \quad . \quad . \quad (1),$$

$$(x-2)(y+1) = xy \quad . \quad . \quad . \quad (2).$$

By transposition and reduction, (1) and (2) become

$$3y - x = 3 \quad . \quad . \quad . \quad (3),$$

$$x - 2y = 2 \quad . \quad . \quad . \quad (4).$$

Eliminating x from (3) and (4),

$$3y - 2y = 5, \text{ or}$$

$$y = 5;$$

$\therefore$ by (4), $\qquad x = 2y + 2 = 10 + 2 = 12.$

3. What fraction is that which becomes equal to $\frac{3}{4}$ when its numerator is increased by 6, and equal to $\frac{1}{2}$ when its denominator is diminished by 2?

Let $x =$ the numerator, and
$y =$ the denominator of the fraction;
then, by the conditions of the problem,

$$\frac{x+6}{y} = \frac{3}{4} \quad . \quad . \quad . \quad (1),$$

$$\frac{x}{y-2} = \frac{1}{2} \quad . \quad . \quad . \quad (2).$$

Clearing of fractions, transposing and reducing,

$$3y - 4x = 24 \quad . \quad . \quad . \quad (3),$$

$$y - 2x = 2 \quad . \quad . \quad . \quad (4).$$

Multiplying (4) by 2, and subtracting the result from (3), we find

$$y = 20;$$

$\therefore$ by (4), $\qquad x = 9.$

Therefore the required fraction is $\frac{9}{20}$.

4. Find two numbers whose sum is a, and whose difference is b.

Let $x =$ the greater number, and
$y =$ the less number;
then, by the conditions of the problem,

$$x + y = a \quad . \quad . \quad . \quad (1),$$
$$x - y = b \quad . \quad . \quad . \quad (2);$$

whence, $$x = \frac{a}{2} + \frac{b}{2},$$

and $$y = \frac{a}{2} - \frac{b}{2}.$$

Since a and b are any numbers whatever, we have the following general principles, by means of which all similar problems can be solved:

1. *The greater of two numbers is found by adding half their difference to half their sum.*

2. *The less of two numbers is found by subtracting half their difference from half their sum.*

5. A and B together possess \$570. If A's money were three times what it really is, and B's five times what it really is, the sum would be \$2350. How much money does each possess?
Ans. A \$250, B \$320.

6. Find two numbers such that if the first be added to four times the second, the sum is 29; and if the second be added to six times the first, the sum is 36. *Ans.* 5 and 6.

7. If A's money were increased by \$36, he would have three times as much as B; but if B's money were diminished by \$5, he would have half as much as A. How much has each?
Ans. A \$42, B \$26.

8. A and B lay a wager of \$10; if A loses, he will have \$25 less than twice as much as B will then have; but if B loses, he will have five-seventeenths of what A will then have. How much money has each? *Ans.* A \$75, B \$35.

9. For \$21, either 32 pounds of tea and 15 pounds of coffee, or 36 pounds of tea and 9 pounds of coffee, can be bought; find the price per pound of each. *Ans.* Tea 50 cts., coffee $33\frac{1}{3}$ cts.

10. A pound of tea and three pounds of sugar cost \$1.20; but if tea were to rise 50 per cent. and sugar 10 per cent., they would cost \$1.56; find the price per pound of each.

Ans. Tea 60 cents, sugar 20 cents.

11. A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days; in what time could each person alone perform the same work?

Ans. A, $14\frac{34}{49}$ days; B, $17\frac{23}{41}$; C, $23\frac{7}{31}$.

12. A and B together can perform a piece of work in a days, A and C together in b days, and B and C together in c days; in what time could each person alone perform the same work?

$$\textit{Ans.} \left\{ \begin{array}{l} \text{A in } \dfrac{2abc}{ac+bc-ab} \text{ days,} \\[2ex] \text{B in } \dfrac{2abc}{ab+bc-ac} \text{ days,} \\[2ex] \text{C in } \dfrac{2abc}{ab+ac-bc} \text{ days.} \end{array} \right.$$

13. A person possesses a certain capital, which is invested at a certain rate per cent. A second person has \$1000 more than the first, and investing his capital one per cent. more advantageously, has an income greater by \$80. A third person has \$1500 more capital than the first, and investing it two per cent. more advantageously, has an income greater by \$150. Find the capital of each person and the rate at which it is invested.

Ans. { Sums at interest, \$3000, \$4000, \$4500.
Rates of interest, 4, 5, 6 per cent.

14. If there were no accidents, it would take half as long to travel the distance from A to B by railroad as by coach; but three hours being allowed for accidental stoppages by the former, the coach will travel all the distance but fifteen miles in the same time; if the distance were two-thirds as great as it is, and the same time allowed for railway stoppages, the coach would take exactly the same time; find the distance from A to B.

Ans. 90 miles.

15. A and B are set to a piece of work which they can finish in thirty days, working together, for which they are to receive

\$64. When the work is half finished, A rests eight days and B four days, in consequence of which the work occupies five and a half days more than it would otherwise have done. How much ought each to receive? *Ans.* A \$22, B \$42.

16. A and B run a mile. First A gives B a start of 44 yards, and beats him by 51 seconds; at the second heat A gives B a start of 1 minute and 15 seconds, and is beaten by 88 yards. In what time can each run a mile?

Ans. A in 5 minutes, B in 6 minutes.

17. A and B start together from the foot of a mountain to go to the summit. A would reach it half an hour before B, but, missing his way, goes a mile and back again needlessly, during which he walks at twice his former pace, and reaches the top six minutes before B. C starts twenty minutes after A and B, and, walking at the rate of two and one-seventh miles per hour, arrives at the summit ten minutes after B. Find A's and B's rates of walking, and the distance from the foot to the summit of the mountain. *Ans.* $2\frac{1}{2}$, 2 miles per hour; distance, 5 miles.

18. A number expressed by two digits is four times the sum of the digits, and if 27 be added to the number the order of the digits will be inverted; find the number.

Let $x =$ the left digit, and

$y =$ the right digit;

then, since x stands in the place of tens, the number will be represented by $10x + y$.

$\therefore$ by the first condition,

$$10x + y = 4(x + y) \quad . \quad . \quad . \quad (1);$$

and by the second condition,

$$10x + y + 27 = 10y + x \quad . \quad . \quad . \quad (2).$$

Solving these equations, we find

$$x = 3, \quad \text{and} \quad y = 6;$$

$\therefore$ $10x + y = 30 + 6 = 36$, the number required.

19. A number is expressed by three digits. The middle digit is equal to twice the left-hand digit, and greater by 3 than the

right-hand digit. If 99 be subtracted from the number, the order of the digits will be inverted; find the number. *Ans.* 241.

20. A number consisting of two digits contains the sum of its digits four times, and their product three times; find the number. *Ans.* 24.

21. A railway train, after traveling for one hour, meets with an accident which delays it one hour, after which it proceeds at three-fifths of its former rate, and arrives at the terminus three hours behind time; had the accident occurred 50 miles further on, the train would have arrived 1 hour and 20 minutes sooner; find the length of the line, and the original rate of the train.

Ans. 100 miles; original rate, 25 miles per hour.

22. A railway train, running from London to Cambridge, meets with an accident which causes it to diminish its speed to $\frac{1}{n}$th of what it was before, in consequence of which it is a hours late. If the accident had occurred b miles nearer Cambridge, the train would have been c hours late. Find the original rate of the train.

Ans. $\dfrac{b(n-1)}{a-c}$ miles per hour.

23. The fore-wheel of a carriage makes six revolutions more than the hind-wheel in going 120 yards. If the circumference of the fore-wheel be increased by one-fourth of its present size, and the circumference of the hind-wheel by one-fifth of its present size, the six will be changed to four. Required the circumference of each wheel. *Ans.* 4 yards and 5 yards.

24. A man starts p hours before a coach, and both travel uniformly; the latter passes the former after a certain number of hours. From this point the coach increases its speed to six-fifths of its former rate, while the man increases his to five-fourths of his former rate, and they continue at these increased rates for q hours longer than it took the coach to overtake the man. They are then 92 miles apart; but had they continued for the same length of time at their original rates, they would have been only 80 miles apart. Show that the original rate of the coach is twice that of the man. Also, if $p + q = 16$, show that the original rate of the coach was 10 miles per hour.

214. SYNOPSIS FOR REVIEW.

CHAPTERS VI—VII. EQUATIONS.

- MEMBERS
  - *First Member.*
  - *Second Member.*
- KINDS
  - *Identical Equation.*
  - *Equation of Condition.*
  - *Numerical Equation.*
  - *Literal Equation.*
- KINDS RESPECTING DEGREE.
  - *Simple.*
  - *Quadratic.*
  - *Higher*
    - Cubic.
    - Biquadratic, etc.
- TRANSFORMATIONS.
  - *Clearing of Fractions.* Rule.
  - *Transposition of Terms.* Rule. Cor.
- SOLUTION OF SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.
- SOLUTION OF PROB.
  - *Statement.*
  - *Solution of Equation.*
  - *Rule.*
- SIMULTANEOUS EQUATIONS.
  - *Solution for two unknown quantities.*
    - Elimination by addition or subtraction. Rule. Sch. 1, 2, 3, 4.
    - Elimination by substitution. Rule.
    - Elimination by comparison. Rule.
    - General Sch.
  - *Rule for groups of Equations containing any number of unknown quantities.*

DISCUSSION OF PROBLEMS LEADING TO SIMPLE EQUATIONS.

215. After a problem has been solved, we may inquire what values the unknown quantities will have, when particular suppositions are made with regard to the given quantities. The determination of these values, and their interpretation, constitute the ***Discussion of the Problem.***

216. INTERPRETATION OF NEGATIVE RESULTS.

1. What number must be added to a number a in order that the sum may be b?

Let $x =$ the required number; then, by the question,

$$a + x = b;$$

whence, $$x = b - a.$$

This is a general solution, a and b being arbitrary quantities.

If $a = 12$, and $b = 25$, we have

$$x = 25 - 12 = 13.$$

But suppose $a = 30$, and $b = 24$; then

$$x = 24 - 30 = -6.$$

How is this negative result to be interpreted?

If we recur to the enunciation of the problem, we see that it now reads thus: What number must be added to 30 in order that the sum may be 24?

Here it is obvious that if the words *added* and *sum* are to retain their arithmetical meanings, the proposed problem is impossible. But we see at the same time that the following problem can be solved: What number must be taken from 30 in order that the *difference* may be 24? The answer to this problem is 6.

The second enunciation differs from the first in these respects: The words *added to* are replaced by *taken from*, and the word *sum* by *difference*.

Hence we may say that, in this example, the *negative* result indicates that the problem, in a strictly arithmetical sense, is impossible; but that a new problem can be formed by appropriate changes in the original enunciation, to which the *absolute value* of the negative result will be the correct answer.

This indicates the convenience of using the word *add*, in Algebra, in a more extensive sense than it has in Arithmetic.

Let x denote a quantity which is to be *added algebraically* to a; then the algebraic sum is $a + x$, whether x be positive or negative.

Hence, the equation $a + x = b$ will be possible algebraically, whether a be greater or less than b.

2. A's age is a years, and B's age is b years; when will A be twice as old as B?

Let $x =$ the required number of years; then, by the question,

$$a + x = 2(b + x);$$

whence, $$x = a - 2b.$$

If $a = 40$ and $b = 15$, then

$$x = 40 - 30 = 10.$$

But suppose $a = 35$ and $b = 20$, then

$$x = 35 - 40 = -5.$$

Here, as in the preceding problem, we are led to inquire into the meaning of the negative result. Now, with the assigned values of a and b, the equation which we have to solve becomes

$$35 + x = 40 + 2x.$$

This equation is impossible, if a strictly arithmetical meaning is to be given to the symbols x and $+$, for 40 is greater than 35, and $2x$ is greater than x. But let us change the enunciation to the following: A's age is 35 years, and B's age is 20 years; when *was* A twice as old as B?

Let $x =$ the required number of years; then, by the question,

$$35 - x = 2(20 - x) = 40 - 2x;$$

whence, $$x = 5.$$

Here again, we may say the *negative* result indicates that the problem, in a strictly arithmetical sense, is impossible; but that a new problem can be formed by appropriate changes in the original enunciation, to which the *absolute value* of the negative result will be the correct answer.

We may observe that the equation corresponding to the new enunciation may be obtained from the original equation by changing x into $-x$.

Suppose the problem had been originally enunciated thus: A's age is a years, and B's age is b years; find the epoch at which A's age is twice that of B.

We cannot tell from the enunciation of the problem whether the required epoch is before or after the present date. If we suppose the required epoch to be x years *after* the present date, we obtain

$$x = a - 2b.$$

If we suppose the required epoch to be x years *before* the present date, we obtain

$$x = 2b - a.$$

If $2b$ is *less* than a, the first supposition is correct, since it leads to a positive value for x; the second supposition is incorrect, since it leads to a negative value for x.

If $2b$ is *greater* than a, the second supposition is correct, since it leads to a positive value for x; the first supposition is incorrect, since it leads to a negative value for x.

Here we may say, then, that a negative result indicates that we made the wrong choice out of two possible suppositions which the problem allowed. But it is important to notice, that when we discover that we have made the wrong choice, it is not necessary to go through the *whole investigation* again, for we can make use of the *result* obtained on the wrong supposition. We have only to take the absolute value of the negative result and place the epoch *before* the present date if we had supposed it after, or *after* the present date if we had supposed it before.

3. A's age is a years, and B's age is b years; when was A twice as old as B?

Let $x =$ the required number of years; then

$$a - x = 2(b - x);$$

whence,

$$x = 2b - a.$$

Now let us *verify* the solution. Substituting $2b - a$ for x, we have

$$a - x = a - (2b - a) = 2a - 2b;$$

and

$$2(b - x) = 2(b - 2b + a) = 2a - 2b.$$

If b is less than a, these results are positive, and there is no arithmetical difficulty. But if b is greater than a, although the

two members are algebraically equal, yet, since they are both *negative* quantities, we cannot say that we have arithmetically verified the solution; and when we recur to the problem, we see that it is impossible if a is less than b; because, if at a given date A's age is less than B's, then A's age never was twice B's, and never will be.

Or, without proceeding to verify the result, we may observe that if b is greater than a, then x is also greater than a, which is inadmissible.

Thus it appears that a problem may be really absurd, and yet the result may not immediately present any difficulty, though when we proceed to examine or verify this result, we may discover an intimation of the absurdity.

The equation

$$a + x = 2(b + x)$$

may be considered as the symbolical expression of the following verbal enunciation:

Suppose a and b to be two quantities; what quantity must be added to each, so that the first sum may be twice the second?

Here the words *quantity*, *sum*, and *added* may all be understood in algebraic senses, so that x, a, and b may be positive or negative.

This algebraic statement includes the arithmetical question about the ages of A and B.

It appears, then, that when we translate a problem into an equation, the same equation may be the symbolical expression of a more comprehensive problem than that from which it was derived.

When the solution of a problem leads to a negative result, and the student wishes to form an analogous problem that shall lead to the corresponding positive result, he may proceed thus:

Change x into $-x$ in the equation that has been obtained, and then, if possible, modify the verbal statement of the problem, so as to make it coincident with the new equation.

We say *if possible*, because in some cases no such verbal modification seems attainable, and the problem may then be regarded

as altogether impossible. To illustrate, take the following problem:

4. A's age is 20 years, and B's age is 30 years; when will the age of A be twice that of B?

Let $x =$ the required number of years; then

$$20 + x = 2(30 + x) = 60 + 2x;$$

whence, $$x = -40.$$

This negative result shows that the epoch is not in the future. Suppose it to be in the past. Changing x into $-x$, the original equation becomes

$$20 - x = 2(30 - x);$$

whence, $$x = 40.$$

This result seems to indicate that 40 years ago—that is, 20 years before A was born, and 10 years before B was born—A was twice as old as B. A manifest absurdity. Hence, the problem is an impossible one.

PRINCIPLES—1. *A negative result may arise from the fact that the problem contains a condition which cannot be arithmetically satisfied; or from the fact that, of two possible suppositions respecting the quality of a quantity, we adopted the wrong one.*

2. *After a problem has been translated into an equation, the quality of any quantity involved will be changed, if we change the sign of the symbol of that quantity.*

PROBLEMS.

1. A father's age is 40 years, and his son's age is 13 years; when will the age of the father be four times that of the son?

Ans. $x = -4$.

Modify the enunciation so that the result shall be $+4$.

2. Find two numbers whose sum is 2 and difference 8.

Ans. -3 and $+5$.

Modify the enunciation so that the result shall be $+3$ and $+5$.

3. The difference of two numbers is 6, and four times the less exceeds five times the greater by 12; find the numbers.

Ans. -42 and -36.

Modify the enunciation so that the result shall be $+42$ and $+36$.

4. Two men, A and B, began trade at the same time, A having three times as much money as B. When A had gained \$400 and B \$150, A had twice as much money as B; how much did each have at first? *Ans.* A was in debt \$300, and B \$100.

Modify as in the preceding examples.

5. There are two numbers whose difference is a; and if three times the greater be added to five times the less, the sum will be b. What are the numbers?

Ans. $\frac{b+5a}{8}$ and $\frac{b-3a}{8}$.

Interpret this result when $a = 24$ and $b = 48$.

6. Two men were traveling on the same road toward Boston, A at the rate of a miles per hour, and B at the rate of b miles per hour. At 6 o'clock A.M. A was at a point m miles from Boston, and at 10 o'clock A.M. B was at a point n miles from Boston. When did A pass B?

Ans. $\frac{m-n-4b}{a-b}$ hours after 6 o'clock A.M.

If $m = 36$, $n = 28$, $a = 5$, and $b = 3$, at what time did A pass B?

ZERO AND INFINITY. FINITE, DETERMINATE, AND INDETERMINATE QUANTITIES.

217. The symbol 0, called ***Nothing,*** or ***Zero,*** is used to denote the absence of value, or to represent a quantity less than any assignable value.

A quantity less than any assignable value is sometimes called an ***Infinitesimal.***

218. The symbol ∞, called ***Infinity,*** is used to represent a quantity greater than any assignable value.

219. ***A Finite Quantity*** is one whose absolute value is comprised between the limits 0 and ∞.

220. ***A Determinate Quantity*** is one which has only a *finite* number of values.

221. ***An Indeterminate Quantity*** is one which has an *infinite* number of values.

INTERPRETATION OF THE FORMS $\frac{A}{0}$, $\frac{A}{\infty}$, $\frac{0}{A}$, $\frac{0}{0}$, $\infty \times 0$, $\frac{\infty}{\infty}$, AND $\infty - \infty$.

222. In order to explain the meaning of these symbols, let us consider the fraction $\frac{A}{B}$.

1. Suppose A to be finite, and to remain unchanged, while B continually decreases; then the value of the fraction $\frac{A}{B}$ will continually increase.

Thus: If $B = A$; then $\frac{A}{B} = 1$;

If $B = \frac{A}{2}$; then $\frac{A}{B} = 2$;

If $B = \frac{A}{10}$; then $\frac{A}{B} = 10$;

If $B = \frac{A}{100}$; then $\frac{A}{B} = 100$;

If $B = \frac{A}{1000000}$; then $\frac{A}{B} = 1000000$.

Hence, it is evident that, when B becomes *less* than any assignable quantity, the fraction $\frac{A}{B}$ will become *greater* than any assignable quantity; hence,

$$\frac{A}{0} = \infty.$$

2. If the denominator B is made to increase continually, while the numerator A remains unchanged, then the value of the frac-

tion $\frac{A}{B}$ will continually decrease; and when the denominator B becomes *greater* than any assignable quantity, the fraction will become *less* than any assignable quantity; hence,

$$\frac{A}{\infty} = 0.$$

3. If the numerator A is made to decrease continually, while the denominator B remains unchanged, then the value of the fraction $\frac{A}{B}$ will continually decrease; and when A becomes less than any assignable quantity, the fraction will also become less than any assignable quantity; hence,

$$\frac{0}{B} = 0.$$

4. Multiplying both members of the equation $\frac{0}{B} = 0$ by B, we have,

$$0 = B \times 0.$$

Dividing both members of this equation by 0, we have,

$$\frac{0}{0} = B.$$

But B is *any* finite quantity; hence $\frac{0}{0}$ is a ***Symbol of Indetermination*** **(221).**

5. Multiplying both members of the equation $\frac{A}{\infty} = 0$ by ∞, we have

$$A = \infty \times 0.$$

But A is *any* finite quantity; hence $\infty \times 0$ is a *symbol of indetermination.*

6. We may place the equation $\frac{0}{0} = B$ under the following form:

$$\frac{\frac{1}{0}}{\frac{1}{0}} = B.$$

But $\frac{1}{0} = \infty$; hence,

$$\frac{\infty}{\infty} = \mathrm{B};$$

therefore $\frac{\infty}{\infty}$ is a *symbol of indetermination.*

7. In the identity $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$ make $a = 0$ and $b = 0$; we then have

$$\frac{1}{0} - \frac{1}{0} = \frac{0}{0}$$

that is,

$$\infty - \infty = \frac{0}{0};$$

hence $\infty - \infty$ is a *symbol of indetermination.*

ORDERS OF INFINITESIMALS AND INFINITIES.

223. ***An Infinitesimal of the First Order*** is one that is infinitely small in comparison with a finite quantity; that is, so small that it may be contained in a *finite* quantity an *infinite* number of times. ***An Infinitesimal of the Second Order*** is one that is infinitely small in comparison with an infinitesimal of the first order. ***An Infinitesimal of the Third Order*** is one that is infinitely small in comparison with an infinitesimal of the second order; and so on.

In order to illustrate, let us consider the continued identity

$$\frac{1}{x} = \frac{x}{x^2} = \frac{x^2}{x^3} = \frac{x^3}{x^4}, \text{ etc.}$$

Let x be an infinitesimal of the first order; then $\frac{1}{x} = \infty$; that is, 1 is infinitely great in comparison with x. Again, since $\frac{1}{x} = \frac{x}{x^2}$, and $\frac{1}{x} = \infty$, it follows that $\frac{x}{x^2} = \infty$; that is, x is infinitely great in comparison with x^2; but x is, by hypothesis, an infinitesimal of the first order; therefore x^2 is an infinitesimal of the second order. In like manner, it may be shown that x^3 is an infinitesimal of the third order, and so on.

224. Infinities are of different orders also. Let x be an infinitesimal of the first order, and A any finite quantity; then,

$$\frac{A}{x} = \infty \ldots (1), \; \frac{A}{x^2} = \infty \ldots (2), \; \frac{A}{x^3} = \infty \ldots (3), \text{ and so on.}$$

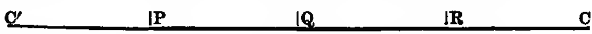
Now the denominator in the first member of (1) is infinitely great in comparison with the denominator in the first member of (2); therefore the second member of (2) is infinitely great in comparison with the second member of (1).

In like manner it may be shown that the ∞ in (3) is infinitely great in comparison with the ∞ in (2); and so on.

225. PROBLEM OF THE COURIERS.

The discussion of the following problem, originally proposed by Clairaut, will serve to illustrate some of the preceding principles:

Two couriers, A and B, were traveling along the same road and in the same direction, namely, from C′ toward C; the former going at the rate of m miles per hour and the latter at the rate of n miles per hour. At 12 o'clock, A was at P, and B was d miles in advance of A. When were the couriers together?



We cannot tell from the enunciation whether the couriers were together before or after 12 o'clock; but in order to effect a statement of the problem, we will suppose the required time to be *after* 12 o'clock. We must then regard time after 12 o'clock as positive, and time before 12 o'clock as negative.

Suppose R to be the point where the couriers met, and Q to be the point where B was at 12 o'clock.

Let $x =$ the required number of hours; then, since A traveled at the rate of m miles per hour, and B at the rate of n miles per hour, we have

$$PR = mx, \quad \text{and} \quad QR = nx.$$

But
$$PR = PQ + QR;$$
$$mx = d + nx;$$

whence, $$x = \frac{d}{m-n}.$$

DISCUSSION.

I. Suppose $$m > n.$$

Under this hypothesis the value of x will be positive, because the denominator $m - n$ is positive. Now, since x is positive, we infer that the couriers were together *after* 12 o'clock.

This conclusion is consistent with the conditions of the problem. For, the supposition is that A was traveling faster than B. A would therefore gain upon B, and overtake him some time after 12 o'clock.

II. Suppose $$m < n.$$

Under this hypothesis the value of x will be negative, because the denominator $m - n$ is negative. This implies that the couriers were together *before* 12 o'clock.

This interpretation, also, agrees with the conditions of the problem under the present hypothesis. For, since $m < n$, B was traveling faster than A; and, as B was in advance of A at 12 o'clock, he must have passed A before that time.

III. Suppose $$m = n.$$

Under this hypothesis we shall have

$$x = \frac{d}{0} = \infty.$$

This result implies that the time to elapse before the couriers are together is greater than any assignable quantity, or infinity; therefore they can never be together.

This interpretation is in accordance with the conditions of the problem under the present hypothesis. For, at 12 o'clock the couriers were d miles apart; and, if $m = n$, they were traveling at equal rates. Hence, they could continue to travel forever without meeting.

IV. Suppose $d = 0$, and $m > n$, or $m < n$.

Then $$x = \frac{0}{m-n} = 0.$$

That is, the time to elapse is nothing. This result implies that the couriers were together at 12 o'clock, and at no other time.

This interpretation is confirmed by the conditions of the problem. For, if $d=0$, then, at 12 o'clock, B must have been with A, at the point P. Moreover, if $m>n$, or $m<n$, the couriers were traveling at different rates, and must have been either approaching or receding from each other at all times except at the moment of passing.

V. Suppose $\qquad d=0, \quad$ and $\quad m=n.$

Then
$$x=\frac{0}{0}.$$

Here the value of x is represented by one of the symbols of indetermination. This result implies that the couriers were together all the time.

This conclusion is evidently confirmed by the conditions of the problem. For, if $d=0$, the couriers were together at 12 o'clock; and, if $m=n$, they were traveling at equal rates, and would never separate.

226. SYNOPSIS FOR REVIEW.

CHAP. VII.—*Con.* **DISCUSSIONS.**

- INTERPRETATION OF NEG. RESULTS.
  - *Principle 1.*
  - *Principle 2.*
- TERMS USED . . . .
  - *Zero.*
  - *Infinitesimal.*
    - First order.
    - Second order.
    - Third order, etc.
  - *Infinity.*
  - *Determinate Quantity.*
  - *Indeterminate Quantity.*
  - *Finite Quantity.*
- SYMBOLS OF INDETERMINATION. $\left\{\frac{0}{0}, \infty\times 0, \frac{\infty}{\infty}, \infty-\infty.\right.$
- SYMBOLS OF ZERO. $\left\{0, \frac{0}{A}, \frac{A}{\infty}.\right.$
- PROBLEM OF COURIERS.

CHAPTER VIII.

VANISHING FRACTIONS.—INDETERMINATE EQUATIONS AND PROBLEMS.—INCOMPATIBLE EQUATIONS.

VANISHING FRACTIONS.

227. ***A Vanishing Fraction*** is one which, on a certain supposition, assumes the form of indetermination. Thus, $\frac{x^3 - a^3}{x - a}$ assumes the form of $\frac{0}{0}$, if $x = a$.

The value of a fraction sometimes reduces to the form of $\frac{0}{0}$, for a particular supposition, in consequence of the existence of a factor common to both terms, which factor reduces to 0 for that supposition. Thus, the fraction $\frac{2a(a-b)}{3b(a-b)}$ reduces to the form of $\frac{0}{0}$, if $a = b$, because the factor $a - b$ becomes 0 in that particular case. But if this factor be canceled, and the supposition that $a = b$ be made afterward, the value of the fraction will be $\frac{2}{3}$. Before deciding, therefore, upon the nature of the symbol $\frac{0}{0}$, we must ascertain whether it results from a factor common to both terms, which reduces to 0 for the supposition made; if it does not, the value of the fraction is *really* indeterminate.

RULE.

I. *Reduce the given fraction to its lowest terms.*

II. *Make the supposition which would cause the original fraction to assume the form of indetermination; the result will be the value of the fraction for that supposition.*

EXAMPLES.

1. Find the value of $\frac{x^4 - y^4}{x^2 - y^2}$, when $x = y$.

Canceling the common factor $x^2 - y^2$, we have,

$$\frac{x^4 - y^4}{x^2 - y^2} = x^2 + y^2;$$

which, when $x = y$, reduces to $2y^2$;

$$\frac{x^4 - y^4}{x^2 - y^2} = 2y^2, \text{ when } x = y.$$

This may be expressed algebraically as follows:

$$\left(\frac{x^4 - y^4}{x^2 - y^2}\right)_{x=y} = 2y^2.$$

2. Find the value of $\left\{\frac{7x(x^4 - 1)}{(1 + x)(x - 1)}\right\}_{x=1.}$ *Ans.* 14.

3. Find the value of $\left\{\frac{2(a - b)^2}{3(a^2 - b^2)}\right\}_{a=b.}$ *Ans.* 0.

4. Find the value of $\left\{\frac{2(a^4 - x^4)}{3(a - x)^2}\right\}_{x=a.}$ *Ans.* ∞.

5. Find the value of $\left(\frac{x^2 - ax}{x^4 - 2ax^3 + 2a^3x - a^4}\right)_{x=a.}$ *Ans.* ∞.

6. Find the value of $\left\{\frac{a + x}{a - x} \times \frac{a^2 - x^2}{(a + x)^2}\right\}_{x=a.}$ *Ans.* 1.

INDETERMINATE EQUATIONS.

228. ***An Indeterminate Equation*** is one in which each of the unknown quantities has an infinite number of values.

229. *A single equation containing two or more unknown quantities is indeterminate.*

Suppose we have an equation containing two unknown quantities, x and y; for example, $2x - 3y = 15$. For every value

which we please to ascribe to one of the unknown quantities we can determine the corresponding value of the other, and thus find as many pairs of values as we please which satisfy the given equation.

Thus, if $y = 1, 2, 3, 4, 5 \ldots.;$

then $x = 9, 10\frac{1}{2}, 12, 13\frac{1}{2}, 15 \ldots.$

Again, suppose we have an equation containing three unknown quantities, x, y, and z; for example, $x + y + 2z = 90$. For every value which we please to ascribe to two of the unknown quantities we can determine the corresponding value of the third, and thus find as many sets of values as we please which satisfy the given equation.

Thus, if $\begin{cases} z = 1,\ 2,\ 3,\ 4,\ 5 \ldots., \\ y = 0,\ 1,\ 2,\ 5,\ 8 \ldots.; \end{cases}$

then $x = 88,\ 85,\ 82,\ 77,\ 72 \ldots.$

A similar course of reasoning is applicable to an equation containing more than three unknown quantities.

230. *Equations are indeterminate if the number of unknown quantities involved exceeds the number of equations.*

For, by eliminating, we can obtain a single equation containing two or more unknown quantities, which is indeterminate (**229**).

Thus, suppose we have the two equations

$$x + y + 2z = 90 \quad . \quad . \quad . \quad (1),$$

$$5x + 2y - 2z = 366 \quad . \quad . \quad . \quad (2).$$

Eliminating z,

$$6x + 3y = 456 \quad . \quad . \quad . \quad (3),$$

which is indeterminate.

231. *An equation containing only one unknown quantity may be indeterminate in consequence of certain relations which subsist between the known quantities.*

If we solve the equation

$$ax + b = cx + d \quad . \quad . \quad . \quad (1),$$

we obtain
$$x = \frac{d - b}{a - c} \quad . \quad . \quad . \quad (2).$$

Now, if $d = b$, and $a = c$,

$$x = \frac{0}{0} \quad . \quad . \quad . \quad (3);$$

hence, under this hypothesis, the value of x is indeterminate.

But, if $d = b$, and $a = c$, (1) becomes

$$cx + b = cx + b \quad . \quad . \quad . \quad (4),$$

which is an *identity*, and may therefore be satisfied for any value of x (**178**).

Here, then, we have *one* unknown quantity and *no* equation; that is, no equation of condition (**179**).

232. *Two equations involving two unknown quantities may be indeterminate in consequence of certain relations which subsist between the known quantities.*

If we solve the equations

$$ax + by = r \quad . \quad . \quad . \quad (1),$$

$$cx + dy = s \quad . \quad . \quad . \quad (2),$$

we obtain
$$x = \frac{dr - bs}{ad - bc} \quad . \quad . \quad . \quad (3),$$

and
$$y = \frac{as - cr}{ad - bc} \quad . \quad . \quad . \quad (4).$$

Now, if
$$dr = bs \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and
$$bc = ad \quad . \quad . \quad . \quad . \quad . \quad (6),$$

then, by multiplying (5) by (6), member by member, and reducing,

$$cr = as \quad . \quad . \quad . \quad . \quad . \quad (7);$$

$\therefore$ (3) and (4) become

$$x=\frac{0}{0}, \quad \text{and} \quad y=\frac{0}{0}.$$

Let us now see what is implied by the relations (5) and (6).

From (5) we have $d=\frac{bs}{r}$, and from (6), $c=\frac{ad}{b}=\frac{as}{r}$. These values of d and c reduce (2) to (1), and we then have only one equation containing two unknown quantities, which is indeterminate.

COR.—The four theorems which have just been demonstrated may be reduced to the following one:

Indetermination arises if the number of unknown quantities exceeds the number of equations.

INDETERMINATE PROBLEMS.

233. ***An Indeterminate Problem*** is one which admits of an infinite number of solutions.

We may often limit the number of solutions by imposing the condition that the values of the unknown quantities shall be *positive integers.*

When an indeterminate problem is expressed in algebraic language, it will be found that the number of unknown quantities exceeds the number of equations.

PROBLEMS.

1. A boy paid 50 cents for some apples and oranges, giving 2 cents each for apples and 10 cents each for oranges. How many of each did he buy?

Let $x=$ the number of apples, and $y=$ the number of oranges; then, by the question,

$$2x+10y=50;$$

whence,
$$x=25-5y.$$

Now, if x and y are to be positive integers, y must be some integer between 0 and 5. Let

$$y=1,\ 2,\ 3,\ 4;$$

then
$$x=20,\ 15,\ 10,\ 5.$$

2. Find two positive integers such that 12 times the one exceeds 13 times the other by 9.

Let $x =$ one of the numbers,

and $y =$ the other;

then, by the question,

$$12x - 13y = 9;$$

whence, $$x = \frac{13y + 9}{12} = y + \frac{y + 9}{12}.$$

Since x and y are to be positive integers, $\frac{y + 9}{12}$ must be an integer. Let

$$\frac{y + 9}{12} = n;$$

whence, $$y = 12n - 9.$$

Let $n = 1, 2, 3, 4 \ldots\ldots$; then $\begin{cases} y = 3, 15, 27, 39 \ldots\ldots \\ x = 4, 17, 30, 43 \ldots\ldots \end{cases}$

3. A man bought 100 animals for \$100; sheep at \$$3\frac{1}{2}$ each, calves at \$$1\frac{1}{3}$, and pigs at \$$\frac{1}{2}$. How many did he buy of each kind?

Let $x =$ the number of sheep,

$y =$ the number of calves,

$z =$ the number of pigs;

then, by the question,

$$x + y + z = 100 \quad . \quad . \quad . \quad (1),$$

$$3\tfrac{1}{2}x + 1\tfrac{1}{3}y + \tfrac{1}{2}z = 100 \quad . \quad . \quad . \quad (2).$$

Combining (1) and (2), eliminating z,

$$18x + 5y = 300 \quad . \quad . \quad . \quad (3);$$

whence, $$y = 60 - \frac{18x}{5} \quad . \quad . \quad . \quad (4).$$

From (1) it is evident, that if x and y are positive integers whose sum is less than 100, z will be a positive integer also.

From (4) it is evident that x must be a multiple of 5, and that $\frac{18x}{5}$ must be less than 60.

Let $\qquad x = 5,\ 10,\ 15;$

then $\qquad y = 42,\ 24,\ 6,$

and $\qquad z = 53,\ 66,\ 79.$

4. The sum of three positive integers is 11; and if the first be multiplied by 3, the second by 5, and the third by 7, the sum of the products will be 57. What are the numbers?

$$\textit{Ans.}\ \begin{cases} x = 4,\ 3,\ 2,\ 1, \\ y = 2,\ 4,\ 6,\ 8, \\ z = 5,\ 4,\ 3,\ 2. \end{cases}$$

5. Divide 200 into two parts, such that if one of them be divided by 6 and the other by 11, the respective remainders may be 5 and 4. *Ans.* 185, 15; 119, 81; 53, 147.

6. Can the equation $4x + 6y = 27$ be solved in positive integers?

7. Find the least number which, being divided by 5, leaves a remainder 3, and divided by 7 leaves a remainder 5. *Ans.* 33.

8. Solve the equation $8x + 13y = 159$ in positive integers.

$$\textit{Ans.}\ \begin{cases} x = 15,\ 2. \\ y = 3,\ 11. \end{cases}$$

INCOMPATIBLE EQUATIONS.

234. ***Incompatible Equations*** are those which cannot be satisfied for the same values of the unknown quantities.

235. Equations are said to be ***Independent*** when they express conditions *essentially different*, and ***Dependent*** when they express the *same* conditions under *different forms*.

Thus, $\left\{\begin{array}{r} x + 3y = 19 \\ 2x + 5y = 33 \end{array}\right\}$ are *independent* equations.

But $\left\{\begin{array}{r} x + 3y = 19 \\ 2x + 6y = 38 \end{array}\right\}$ are *dependent* equations, since the second may be obtained from the first by multiplying both members by 2.

236. *If the number of independent equations exceeds the number of unknown quantities, these equations may be incompatible.*

Let us consider the three equations

$$x + y = 8 \quad . \quad . \quad . \quad (1),$$
$$x - y = 2 \quad . \quad . \quad . \quad (2),$$
$$\frac{x}{y} = 2 \quad . \quad . \quad . \quad (3).$$

Combining (1) and (2), we find $x = 5$, and $y = 3$; but these values will not satisfy (3). In like manner it may be shown that the values which satisfy (2) and (3) do not satisfy (1), and that the values which satisfy (1) and (3) do not satisfy (2).

237. *If the number of independent equations exceeds the number of unknown quantities, such relations between the known quantities can be found as will make the equations compatible.*

Let us consider the equations

$$x + y = s \quad . \quad . \quad . \quad (1),$$
$$x - y = d \quad . \quad . \quad . \quad (2),$$
$$x = ay \quad . \quad . \quad . \quad (3).$$

Combining (1) and (2), we find

$$x = \frac{s + d}{2},$$

$$y = \frac{s - d}{2}.$$

Substituting these values in (3),

$$\frac{s + d}{2} = a\left(\frac{s - d}{2}\right);$$

whence, $$a = \frac{s + d}{s - d} \quad . \quad . \quad . \quad (4).$$

If the relation expressed by (4) subsists, (1), (2), and (3) will be compatible. Thus, the equations,

$$x + y = 9,$$
$$x - y = 3,$$
$$x = 2y,$$

are compatible, for

$$2 = \frac{9 + 3}{9 - 3}.$$

Cor.—In order that a problem may be *determinate*, the conditions must furnish as many independent equations as there are unknown quantities.

Sch. 1.—When a problem contains more conditions than are necessary for determining the values of the unknown quantities, those that are unnecessary are termed *redundant.*

Sch. 2.—A problem, from which incompatible equations are deduced, is called an *impossible problem.* Such a problem is said to involve *incompatible conditions.*

238. SYNOPSIS FOR REVIEW.

CHAPTER VIII.
- Vanishing Fractions. *Investigation.* Rule.
- Theorems relating to Indeterminate Equations.
  - **229.**
  - **230.**
  - **231.**
  - **232.**
  - *Reduction of the* four *theorems to* one.
- Indeterminate Problems. *No. solutions limited, how.*
- Incompatible Equations.
  - *Terms used.*
    - Dependent Equations.
    - Independent Equations.
  - *Theorems.*
    - **236.**
    - **237.** *Cor.; Sch. 1, 2.*

CHAPTER IX.

INEQUALITIES.

239. ***An Inequality*** consists of two expressions connected by the sign of inequality.

The First Member of an inequality is the expression on the left of the sign of inequality, and the *Second Member* is the expression on the right of the sign.

240. Two inequalities *subsist in the same sense* when the first member is the greater in each, or the less in each. Thus, $5 > 3$ and $7 > 4$ subsist in the same sense.

Two inequalities *subsist in a contrary sense* when the first member is the greater in one, and the less in the other. Thus, $5 > 1$ and $4 < 8$ subsist in a contrary sense.

241. *If the same quantity be added to, or subtracted from, each member of an inequality, the resulting inequality will subsist in the same sense.*

For, suppose $a > b$; then $a - b$ is positive (**118**). Again, since $a \pm c - (b \pm c) = a - b$, it follows that $a \pm c - (b \pm c)$ is positive;

$$\therefore \qquad a \pm c > b \pm c.$$

Cor. 1.—The rule for the transposition of terms in equations is applicable to inequalities. Thus, if

$$a^2 + b^2 > 2ab + c^2;$$

then $$a^2 + b^2 - 2ab > 2ab - 2ab + c^2,$$

or $$a^2 - 2ab + b^2 > c^2;$$

Cor. 2.—If an equation be added to an inequality, member to member, or subtracted from it, member from member, the resulting inequality will subsist in the same sense. Thus, if

$$a > b,$$

and $$x = y,$$

then $$a \pm x > b \pm y.$$

242. *If an inequality be subtracted from an equation, member from member, the sign of inequality will be reversed.*

For, suppose $$x = y,$$

and $$a > b;$$

then $$x - a - (y - b) = b - a,$$

and $b - a$ is negative;

$$\therefore \qquad x - a < y - b.$$

243. *If both members of an inequality be multiplied or divided by the same positive quantity, the resulting inequality will subsist in the same sense.*

For, suppose m to be positive, and

$$a > b;$$

then, since $a - b$ is positive, $m(a - b)$ and $\frac{1}{m}(a - b)$ are positive;

$$\therefore \qquad ma > mb \quad \text{and} \quad \frac{a}{m} > \frac{b}{m}.$$

244. *If both members of an inequality be multiplied or divided by the same negative quantity, the resulting inequality will subsist in a contrary sense.*

For, suppose m to be negative, and

$$a > b;$$

then, since $a - b$ is positive, $m(a - b)$ and $\frac{1}{m}(a - b)$ are negative;

$$\therefore \qquad ma < mb \quad \text{and} \quad \frac{a}{m} < \frac{b}{m}.$$

COR.—If the signs of all the terms of an inequality be changed, the sign of inequality must be reversed. For changing the signs of all the terms is equivalent to multiplying each member by -1.

245. *If two or more inequalities subsisting in the same sense be added, member to member, the resulting inequality will subsist in the same sense as the given inequalities.*

For, if $\qquad a > b, \quad a' > b', \quad \text{and} \quad a'' > b''$,

then $\quad a - b, \quad a' - b', \quad$ and $\quad a'' - b'' \quad$ are positive; therefore,

$a - b + a' - b' + a'' - b'', \quad$ or $\quad a + a' + a'' - (b + b' + b'')$,

is positive;

$$\therefore \qquad a + a' + a'' > b + b' + b''.$$

SCH.—If one inequality be *subtracted* from another subsisting in the same sense, the result will not always be an inequality subsisting in the same sense as the given inequalities, or an inequality at all.

Take the two inequalities

$$4 < 7,$$

and $$2 < 3.$$

By subtraction, $$2 < 4.$$

Here the result is an inequality subsisting in the *same* sense as the given inequalities.

But take $$9 < 10,$$

and $$6 < 8.$$

By subtraction, $$3 > 2.$$

Here the result is an inequality subsisting in a sense *contrary* to that of the given inequalities.

Again, take $$9 < 10,$$

and $$6 < 7.$$

By subtraction, $$3 = 3.$$

Here the result is an equation.

246. ***The Reduction*** of an inequality consists in transforming it in such a manner that the unknown quantity may stand alone as one of its members. The other member will then denote one *limit* of the unknown quantity.

EXAMPLES.

Reduce the following inequalities:

1. $$\frac{x}{2}+\frac{2x}{5}>\frac{3x}{4}+\frac{9}{4}.$$

Multiplying both members by 20,

$$10x+8x>15x+45;$$

transposing and reducing,

$$3x>45;$$

whence,

$$x>15.$$

2. $5x>\frac{3x}{2}+14.$ *Ans.* $x>4.$

3. $\frac{2x}{5}-\frac{2x}{3}>\frac{2x}{5}-2.$ *Ans.* $x<3.$

4. $\frac{5x}{8}+\frac{5}{4}<\frac{11}{6}+\frac{7x}{12}.$ *Ans.* $x<14.$

247. If there be given an inequality and an equation, involving two unknown quantities, a limit of each unknown quantity may be found by elimination.

EXAMPLES.

1. Find a limit for x and y in the following groups:

1. $$\begin{cases} 2x+5y>16 & \quad . \quad . \quad . \quad (1), \\ 2x+y=12 & \quad . \quad . \quad . \quad (2). \end{cases}$$

Subtracting (2) from (1),

$$4y>4;$$

whence,

$$y>1.$$

If we substitute 1 for y in (2), the first member will be made less than the second; hence,

$$2x + 1 < 12;$$

whence,

$$x < 5\tfrac{1}{2}.$$

2. $\left\{\begin{array}{l} 2x + 4y > 30 \\ 3x + 2y = 31 \end{array}\right\}.$ *Ans.* $x < 8,\ y > 3\tfrac{1}{2}.$

3. $\left\{\begin{array}{l} 5x - 3y < 15 \\ 9x + 2y = 46 \end{array}\right\}.$ *Ans.* $x < 4\tfrac{20}{37},\ y > 2\tfrac{21}{37}.$

4. $\left\{\begin{array}{l} 7x - 10y < 59 \\ 4x + \ \ 5y = 68 \end{array}\right\}.$ *Ans.* $x < 13,\ y > 3\tfrac{1}{5}.$

5. $\left\{\begin{array}{l} 5x + 3y > 121 \\ 7x + 4y = 168 \end{array}\right\}.$ *Ans.* $x < 20,\ y > 7.$

6. $\left\{\begin{array}{l} \dfrac{x-4}{8} - \dfrac{y-10}{6} > 1 \\[2ex] \dfrac{3x-24}{4} + \dfrac{x-y}{2} = 13 \end{array}\right\}.$

248. SYNOPSIS FOR REVIEW.

CHAPTER IX. INEQUALITIES.

- MEMBERS.
- SUBSISTING IN THE SAME SENSE.
- SUBSISTING IN A CONTRARY SENSE.
- THEOREMS . . . .
  - **241.**—*Cor. 1, 2.*
  - **242.**
  - **243.**
  - **244.**—*Cor.*
  - **245.**—*Sch.*
- REDUCTION.
- COMBINATION OF AN EQUATION WITH AN INEQUALITY.

CHAPTER X.

INVOLUTION AND EVOLUTION.

INVOLUTION.

249. ***A Power*** of a quantity is the product of factors each of which is equal to that quantity. A quantity is said to be *raised* or *involved* when any power of it is found.

250. ***Involution*** is the process of raising a given quantity to any required power.

251. ***The Base*** or ***Root*** of a power is one of the equal factors of the power, and the ***Degree*** of a power is equal to the number of times the base is used as a factor to produce the power. Thus, a^3 is the *third power* or cube of a, and a is the base of a^3.

252. ***The Exponent of the Power*** is the exponent which indicates the power to which the given quantity is to be raised.

253. ***A Perfect Power*** of the n^{th} degree is a quantity which can be resolved into n equal factors. Thus, $a^2 - 2ab + b^2$ is a perfect power of the second degree.

254. ***An Imperfect Power*** of the n^{th} degree is a quantity which cannot be resolved into n equal factors. Thus, $a^2 - b^2$ is an imperfect power.

A quantity may be a perfect power of one degree, but an imperfect power of another degree. Thus, $a^2 - 2ab + b^2$ is a perfect power of the second degree, but an imperfect power of the third degree.

255. *The Sign of the Power.*—Any power of a positive quantity is positive; for, when all the factors are positive, the product is positive. If the quantity to be involved is negative, the *even* powers will be positive, and the *odd* powers will be negative.

For, $(-a)(-a)=a^2$, $(-a)(-a)(-a)=a^2(-a)=-a^3$, $(-a)(-a)(-a)(-a)=-a^3(-a)=a^4$, and so on.

256. *The n^{th} Power of a Product.*—It follows, from Art. **249**, that $(ab)^n = ab \times ab \times ab \ldots$ to n factors $= a \times a \times a \ldots$ to n factors $\times b \times b \times b \ldots$ to n factors $= a^n b^n$. In like manner, $(abc)^n = a^n b^n c^n$, and $(abc \ldots k)^n = a^n b^n c^n \ldots k^n$.

$\therefore$ *The n^{th} power of the product of any number of factors is equal to the product of the n^{th} powers of those factors*

Again,
$$(a^n)^2 = a^n \times a^n = a^{n+n} = a^{2n},$$
$$(a^n)^3 = a^n \times a^n \times a^n = a^{n+n+n} = a^{3n},$$
$$(a^n)^m = a^{mn}.$$

$\therefore$ *If the n^{th} power of a quantity be raised to the m^{th} power, the result will be the mn^{th} power of that quantity.*

257. *The Coefficient of the Power.*—Since $(5a)^3 = 5a \times 5a \times 5a = 5^3a^3 = 125a^3$, and $(5a)^n = 5^n a^n$, it follows that

The coefficient of the n^{th} power of a quantity is the n^{th} power of the coefficient of that quantity.

258. To find any power of a monomial.

RULE.

Raise the numerical coefficient to the required power, and write after the result all the letters of the given monomial, giving to each an exponent equal to the product of its original exponent by the exponent of the power.

EXAMPLES.

1. Find the 3d power of $2a^2b^5c$. *Ans.* $8a^6b^{15}c^3$.
2. Find the 6th power of the 5th power of a^3bc^2. *Ans.* $a^{90}b^{30}c^{60}$.

3. Find the 5th power of $-abx^n$. *Ans.* $-a^5b^5x^{5n}$.

4. Find the 4th power of $-a^2b^3cd^2$. *Ans.* $a^8b^{12}c^4d^8$.

5. Find the 5th power of $3a^2x^2$. *Ans.* $243a^{10}x^{10}$.

6. Find the 7th power of $a^2b^3c\,(-x^2y^3z^4)$.
Ans. $-a^{14}b^{21}c^7x^{14}y^{21}z^{28}$.

7. Find the 3d power of $(-ab^4c^2)(-a^2b^3c)$. *Ans.* $a^9b^{21}c^9$

8. Find the 5th power of $(-a)^2(-b)^3(-c)^4$.
Ans. $-a^{10}b^{15}c^{20}$.

9. Find the m^{th} power of $-a^2b^5c^3$ when m is an even positive integer. *Ans.* $a^{2m}b^{5m}c^{3m}$.

10. Find the m^{th} power of $-(abc)^2$ when m is a positive integer. *Ans.* $\pm a^{2m}b^{2m}c^{2m}$.

259. To find any power of a polynomial.

RULE.

Find the product of as many factors, each of which is equal to the given polynomial, as there are units in the exponent of the required power.

ILLUSTRATIONS.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad\;+ab+b^2 \\ \hline a^2+2ab+b^2, \end{array} \qquad \begin{array}{l} a^2+2ab+b^2 \\ a+b \\ \hline a^3+2a^2b+ab^2 \\ \quad\;+a^2b+2ab^2+b^3 \\ \hline a^3+3a^2b+3ab^2+b^3, \end{array} \qquad \begin{array}{l} a^3+3a^2b+3ab^2+b^3 \\ a+b \\ \hline a^4+3a^3b+3a^2b^2+ab^3 \\ \quad\;+a^3b+3a^2b^2+3ab^3+b^4 \\ \hline a^4+4a^3b+6a^2b^2+4ab^3+b^4. \end{array}$$

$$\therefore \quad \begin{aligned} (a+b)^2 &= a^2+2ab+b^2, \\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3, \\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4. \end{aligned}$$

In like manner it may be shown that

$$\begin{aligned} (a-b)^2 &= a^2-2ab+b^2, \\ (a-b)^3 &= a^3-3a^2b+3ab^2-b^3, \\ (a-b)^4 &= a^4-4a^3b+6a^2b^2-4ab^3+b^4. \end{aligned}$$

COR. 1.—Since $(a^m)^n = (a^n)^m$ **(256)**, we may reach the same result by different processes of involution. For example, we may find the sixth power of $a + b$ by repeated multiplication by $a+b$; or we may first find the cube of $a + b$, and then square the result; or we may first find the square of $a + b$, and then cube the result.

The work may sometimes be abridged by using the principle expressed by the equation $a^m a^n = a^{m+n}$. Thus, we may find the fifth power of $a + b$ by multiplying the cube of $a + b$ by the square of $a + b$.

COR. 2.—It may be shown by actual multiplication, that

$(a+b+c)^2 = a^2 + 2a(b+c) + b^2 + 2bc + c^2$, and

$(a+b+c+d)^2 = a^2 + 2a(b+c+d) + b^2 + 2b(c+d) + c^2 + 2cd + d^2$.

Hence, we may infer that

The square of any polynomial is equal to the sum of the squares of its terms, together with twice the sum of the products obtained by multiplying each term by the sum of all the terms which follow it.

EXAMPLES.

1. Find the square of $(a - b + c)$.
Ans. $a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$.

2. Find the square of $1 + x + x^2 + x^3$.
Ans. $1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6$.

3. Find $(1 - 2x + 3x^2)^2$.
Ans. $1 - 4x + 10x^2 - 12x^3 + 9x^4$.

4. Find $(a + b - c)^3$.
Ans. $a^3 + b^3 - c^3 + 3a^2(b-c) + 3b^2(a-c) + 3c^2(a+b) - 6abc$.

5. Find $(1 + 2x + x^2)^3$.
Ans. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.

6. Find $(a + b)^6$.
Ans. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

260. **To find any power of a fraction.**

$$\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \text{ to } n \text{ factors} = \frac{a \times a \times \cdot a \ldots \text{ to } n \text{ factors}}{b \times b \times b \ldots \text{ to } n \text{ factors}} = \frac{a^n}{b^n}.$$

Hence,

The n^{th} *power of a fraction is a fraction whose numerator is the* n^{th} *power of the given numerator, and whose denominator is the* n^{th} *power of the given denominator.*

EXAMPLES.

1. Find $\left(\frac{a^2b^3c^4}{d^5e^2m}\right)^2$. *Ans.* $\frac{a^4b^6c^8}{d^{10}e^4m^2}$.

2. Find $\left[\left(\frac{-a^2b^3}{c^3d}\right)^2\right]^3$. *Ans.* $\frac{a^{12}b^{18}}{c^{18}d^6}$.

3. Find $\left(\frac{a+b}{a-b}\right)^2$. *Ans.* $\frac{a^2+2ab+b^2}{a^2-2ab+b^2}$.

4. Find $\left(\pm\frac{a^2b^5c}{4d^3z^4}\right)^4$. *Ans.* $\frac{a^8b^{20}c^4}{256d^{12}z^{16}}$.

5. Show that $\left(\frac{27a^4-18a^2b^2-b^4}{8ab}\right)^2 + \frac{(9a^2-b^2)^3(b^2-a^2)}{(8ab)^2} = b^2$.

EVOLUTION.

261. Let n be any positive integer; then

The n^{th} ***Root*** of any *given quantity* is a quantity the n^{th} power of which is equal to the *given quantity.*

262. ***Evolution,*** or ***The Extraction of Roots,*** is the process of finding any root of a given quantity. Evolution is the converse of involution.

263. ***The Sign of the Root.***—If the *index* of the root to be extracted be an *odd* number, the sign of the root will be the same as the sign of the given quantity (**255**). Thus, $\sqrt[3]{a^3} = a$, and $\sqrt[3]{-a^3} = -a$.

If the *index* of the root to be extracted be an *even* number, and the given quantity be positive, the root may be either positive or negative. Thus, $\sqrt{a^2} = \pm a$.

264. If the index of the root to be extracted be an *even* number, and the given quantity be negative, the root cannot be extracted; because no quantity raised to an even power can produce a negative result (**255**). The *indicated* even root of a negative quantity is called an ***Imaginary Quantity.*** Thus, $\sqrt{-9}$, $\sqrt{-a^2}$, and $\sqrt{-(a+b)^2}$ are imaginary quantities.

265. To find any root of a monomial.

$$(3a^mb^r)^n = 3^na^{mn}b^{rn} \quad (\mathbf{258});$$

$$\sqrt[n]{3^na^{mn}b^{rn}} = 3a^mb^r \quad (\mathbf{261}).$$

RULE.

Extract the required root of the numerical coefficient, and write after the result all the letters of the given monomial, giving to each an exponent equal to the quotient obtained by dividing its original exponent by the index of the root.

EXAMPLES.

1. Find $\sqrt[3]{8a^6b^{15}c^3}$. *Ans.* $2a^2b^5c$.
2. Find $\sqrt[6]{a^{90}b^{30}c^{60}}$. *Ans.* $\pm a^{12}b^5c^{10}$.
3. Find $\sqrt[5]{-a^5b^5x^{5n}}$. *Ans.* $-abx^n$.
4. Find $\sqrt[4]{a^8b^{12}c^4d^8}$. *Ans.* $\pm a^2b^3cd^2$.
5. Find $\sqrt[m]{a^{2m}b^{5m}c^{3m}}$, when m is an *even* positive integer. *Ans.* $\pm a^2b^5c^3$.
6. Find $\sqrt{-a^4b^6c^2}$.

266. To find the square root of a polynomial.

Since the square root of $a^2 + 2ab + b^2$ is $a + b$, we may be led to a general rule for the extraction of the square root of a

polynomial by observing in what manner $a + b$ may be derived from $a^2 + 2ab + b^2$.

$$\begin{array}{r|l}
 & a^2 + 2ab + b^2 \;\big|\; a+b \\
 & a^2 \\ \hline
2a + b & 2ab + b^2 \\
 & \underline{2ab + b^2}
\end{array}$$

Arrange the terms according to the descending powers of a; then the square root of the first term, a^2, is a, which is the first term of the required root. Subtracting its square, that is, a^2, from the given polynomial, we obtain the remainder $2ab + b^2$. Dividing $2ab$ by $2a$, we obtain b, which is the second term of the root. Multiplying $2a + b$ by b, and subtracting the product from the first remainder, we obtain 0 for the second remainder; hence $a + b$ is the required root.

When the root contains three or more terms, it may be found by a similar process. Thus,

$$\begin{array}{r|l}
 & a^2 + 2ab + b^2 + 2(a + b)c + c^2 \;\big|\; a + b + c \\
 & a^2 \\ \hline
2a + b & 2ab + b^2 \\
 & 2ab + b^2 \\ \hline
2(a + b) + c & 2(a + b)c + c^2 \\
 & \underline{2(a + b)c + c^2}
\end{array}$$

The first term of the required root is a. Subtracting a^2 from the given polynomial, we obtain the remainder $2ab + b^2 + 2(a + b)c + c^2$. Dividing the first term of this remainder by $2a$, we obtain the second term of the root. Multiplying $2a + b$ by b, and subtracting the product from the first remainder, we obtain the second remainder, $2(a + b)c + c^2$. Dividing the first term of the second remainder by $2a$, we obtain the third term of the root. Multiplying $2(a + b) + c$ by c, and subtracting the product from the second remainder, we obtain 0 for the third remainder; hence $a + b + c$ is the required root.

We call $2a$ the *partial* divisor, $2a + b$ the first *complete* divisor, and $2(a + b) + c$ the second *complete* divisor.

RULE.

I. *Arrange the given polynomial according to the powers of one of its letters.*

II. *Extract the square root of the first term; the result will be the first term of the required root. Subtract the square of this term from the given polynomial.*

III. *Divide the first term of the remainder by twice the first term of the root, and annex the result to the first term of the root and also to the divisor; then multiply the divisor thus completed by the second term of the root, and subtract the product from the first remainder.*

IV. *Take twice the sum of the first and second terms of the root for a second divisor. Divide the first term of the second remainder by the first term of the second divisor, and annex the result to the part of the root already found and also to the second divisor; then multiply the divisor thus completed by the third term of the root, and subtract the product from the second remainder.*

V. *If the required root contains additional terms, proceed in like manner until all the terms are found.*

COR. 1.—If the first term of the arranged polynomial is not a perfect square, or if the first term of any arranged remainder is not divisible by twice the first term of the root, the exact square root cannot be found.

COR. 2.—All *even* roots admit of the double sign **(263)**; hence the square root of $a^2 + 2ab + b^2$ is $-(a+b)$, as well as $a + b$. In fact, the first term in the root, which we found by extracting the square root of a^2, might have been $-a$; and by using this we should have obtained $-b$ for the second term of the root

EXAMPLES

Find the square root of each of the following expressions:

1. $4x^4 - 12x^3 + 5x^2 + 6x + 1$. *Ans.* $\pm(2x^2 - 3x - 1)$.
2. $1 + 4x + 10x^2 + 12x^3 + 9x^4$. *Ans.* $\pm(1 + 2x + 3x^2)$.

3. $9x^4 + 12x^3 + 22x^2 + 12x + 9$. *Ans.* $\pm(3x^2 + 2x + 3)$.

4. $9a^2 + 12ab + 4b^2 + 6ac + 4bc + c^2$.
Ans. $\pm(3a + 2b + c)$.

5. $4a^4 - 12a^3 + 25a^2 - 24a + 16$. *Ans.* $\pm(2a^2 - 3a + 4)$.

6. $16x^4 - 16abx^2 + 16b^2x^2 + 4a^2b^2 - 8ab^3 + 4b^4$.

7. $x^6 - 4x^5 + 10x^4 - 20x^3 + 25x^2 - 24x + 16$.

8. $81x^4 - 432x^3 + 864x^2 - 768x + 256$.

9. $(a - b)^4 - 2(a^2 + b^2)(a - b)^2 + 2(a^4 + b^4)$.

10. $a^4 + b^4 + c^4 + d^4 - 2a^2(b^2 + d^2) - 2b^2(c^2 - d^2) + 2c^2(a^2 - d^2)$.

267. ***When a Trinomial is a Perfect Square.***— Since $(a \pm b)^2 = a^2 \pm 2ab + b^2$, it follows that a trinomial is a perfect square, if two of its terms are squares, and the other term is twice the product of the square roots of these two.

When a trinomial is a perfect square, its square root may be found by extracting the square root of each of the square terms and connecting the results by the sign of the other term.

EXAMPLES.

1. Extract the square root of $4x^2 - 12xy + 9y^2$.

This is a perfect square, because $4x^2$ and $9y^2$ are squares, and $12xy$ is equal to twice the product of the square roots of these terms. The square root of $4x^2$ is $2x$, and the square root of $9y^2$ is $3y$. Connecting these results by the sign of the term $12xy$, we obtain $2x - 3y$, or $3y - 2x$.

2. Extract the square root of $x^2 + 6xy + 9y^2$.
Ans. $\pm(x + 3y)$.

3. Extract the square root of $9a^2 + 25b^2 - 30ab$.
Ans. $\pm(3a - 5b)$.

4. Extract the square root of $9a^2 - 12ab + 16b^2$.

268. *An expression which, in its simplest form, is a binomial, cannot be a perfect square.* For the square of a monomial is a monomial, and the square of any polynomial contains at least three terms.

269. To find the square root of a number.

PRINCIPLES.—1. *The square of a number consisting of tens and units is equal to the sum of the squares of the tens and the units increased by twice their product.* Thus,

$$78^2 = (70+8)^2 = 70^2 + 2 \times 70 \times 8 + 8^2 = 6084.$$

2. *The square of a number expressed by a single figure contains no figure of a higher order than tens.*

For 9 is the largest number which can be expressed by a single figure, and $9^2 = 81$.

3. *The square of tens contains no significant figure of a lower order than hundreds, nor of a higher order than thousands.*

Thus, $10^2 = 100$, and $90^2 = 8100$.

4. *The square of a number contains twice as many figures as the number, or twice as many less one.* Thus,

$$\begin{array}{rlcrl} 1^2 = & 1, & \qquad & 10^2 = & 100, \\ 9^2 = & 81, & & 100^2 = & 10000, \\ 99^2 = & 9801, & & 1000^2 = & 1000000. \end{array}$$

Hence,

5. *If a number be separated into periods of two figures each, beginning at units' place, the number of periods will be equal to the number of figures in the square root of that number.* Thus, there are two figures in the square root of 43,56.

1. Let it be required to extract the square root of 4356.

$$\begin{array}{rl} & \quad\; a \quad\;\; b \\ & 43,56(60+6=66 \\ a^2 = & 36\,00 \\ 2a+b = 120+6 = 126) & \overline{7\,56} \\ 2ab+b^2 = & \underline{7\,56} \end{array}$$

Let a represent the value of the first figure of the root, and b that of the second figure.

Since 56 cannot be a part of the square of the tens (PRIN. 3), a must be the greatest multiple of ten whose square is less than 4300; this is found to be 60. Subtracting a^2,

that is, 3600, from the given number, we find the remainder to be 756. This remainder consists of *twice the product of the tens by the units, and the square of the units* (PRIN. 1). But, since the product of tens by units cannot be of a lower order than tens, the last figure, 6, cannot be a part of twice the product of the tens by the units; this double product must therefore be found in the part 750.

Now, if we double the tens and divide 750 by the result, the quotient, 6, will be the units' figure of the root, or a figure greater than the units' figure. This quotient figure cannot be too small, for the part 750 is at least equal to twice the product of the tens by the units; but it may be too large, for 750, besides the double product of the tens by the units, may contain tens arising from the square of the units (PRIN. 2).

To ascertain if the quotient, 6, is correct, we add it to 120 and multiply the sum by 6. Subtracting the product from 756, we find the remainder to be 0; hence 66 is the required square root.

2. If the square root contains more than two figures, it may be found by a similar process, as in the following example, where it will be seen that the partial divisor at each step is obtained by doubling that part of the root already found. The letters show how the different steps correspond to those of the algebraic process in Art. **266.**

$$
\begin{array}{rl}
 & \;a \quad\;\; b \quad c \\
 & 18,66,24(400+30+2=432 \\
a^2= & 16\ 00\ 00 \\
2a+b=800+30=830) & 26624=2ab+b^2+2ac+2bc+c^2 \\
 & 24900=2ab+b^2 \\
2(a+b)+c=800+60+2=862) & 1724=2ac+2bc+c^2 \\
 & 1724=2ac+2bc+c^2
\end{array}
$$

For the sake of brevity, the ciphers on the right are usually omitted; thus,

$$
\begin{array}{rl}
 & 43,56(66 \\
 & 36 \\
126) & 756 \\
 & 756
\end{array}
\qquad\qquad
\begin{array}{rl}
 & 18,66,24(432 \\
 & 16 \\
83) & 266 \\
 & 249 \\
862) & 1724 \\
 & 1724
\end{array}
$$

RULE.

I. *Separate the given number into periods of two figures each, beginning at the units' place.*

II. *Find the greatest number whose square is contained in the period on the left; this will be the first figure in the root. Subtract the square of this figure from the period on the left, and to the remainder annex the next period to form a dividend.*

III. *Divide this dividend, omitting the figure on the right, by double the part of the root already found, and annex the quotient to that part, and also to the divisor; then multiply the divisor thus completed by the figure of the root last obtained, and subtract the product from the dividend.*

IV. *If there are more periods to be brought down, continue the operation in the same manner as before.*

EXAMPLES.

Find the square root of each of the following numbers:

1. 177241. *Ans.* 421.
2. 4334724. *Ans.* 2082.
3. 14356521. *Ans.* 3789.
4. 17.338896. *Ans.* 4.164.
5. 2.5. *Ans.* 1.5811+.

270. **To find the cube root of a polynomial.**

$(a+b+c)^3 = a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3.$

Let us now find the root $a+b+c$ from its cube.

$$
\begin{array}{r|l|l}
 & a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3 & a+b \\
 & a^3 & \quad +c \\
\hline
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 & \\
 & \underline{3a^2b+3ab^2+b^3} & \\
\hline
3(a+b)^2+3(a+b)c+c^2 & 3(a+b)^2c+3(a+b)c^2+c^3 & \\
 & \underline{3(a+b)^2c+3(a+b)c^2+c^3} &
\end{array}
$$

The first term of the root is obtained by extracting the cube root of a^3. Subtracting a^3 from the given polynomial, and dividing the first term of the remainder by $3a^2$, we obtain the second term of the root. Multiplying $3a^2 + 3ab + b^2$ by b, and subtracting the product from the first remainder, we obtain the second remainder. Dividing the first term of the second remainder by $3a^2$, we obtain the third term of the root. Multiplying $3(a+b)^2+3(a+b)c+c^2$ by c, and subtracting the product from the second remainder, we obtain 0 for the remainder.

RULE.

I. *Arrange the given polynomial according to the powers of one of its letters; then the cube root of the first term will be the first term of the root. Subtract the cube of the first term of the root from the given polynomial.*

II. *Divide the first term of the remainder by the partial divisor, which is three times the square of the first term of the root; the quotient will be the second term of the root.*

III. To *the partial divisor add three times the product of the first and second terms of the root, also the square of the second term; the result will be the first complete divisor.*

IV. *Multiply the complete divisor by the second term of the root, and subtract the product from the first remainder.*

V. *Divide the first term of the second remainder by the partial divisor, which is three times the square of the first term of the root; the quotient will be the third term of the root.*

VI. *To three times the square of the sum of the first and second terms of the root, add three times the product of the sum of the first and second terms by the third, also the square of the third term; the result will be the second complete divisor.*

VII. *Multiply the second complete divisor by the third term of the root, and subtract the product from the second remainder.*

VIII. *If the required root contains additional terms, proceed in like manner until all the terms are found.*

COR.—We may dispense with the complete divisor, if, after each time that we find a new term of the root, we subtract the cube of the sum of the terms already found from the given polynomial.

EXAMPLES.

Find the cube root of each of the following expressions:

1. $x^3 + 6x^2y + 12xy^2 + 8y^3$. *Ans.* $x + 2y$.

2. $x^3 + 12x^2 + 48x + 64$. *Ans.* $x + 4$.

3. $a^3 - 9a^2 + 27a - 27$. *Ans.* $a - 3$.

4. $8a^3 - 36a^2b + 54ab^2 - 27b^3$. *Ans.* $2a - 3b$.

5. $x^6 + 6x^5 - 40x^3 + 96x - 64$. *Ans.* $x^2 + 2x - 4$.

6. $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1$. *Ans.* $a^2 + 2a + 1$.

7. $x^6 - 12x^5 + 54x^4 - 112x^3 + 108x^2 - 48x + 8$. *Ans.* $x^2 - 4x + 2$.

8. $a^6 - 3a^5b + 6a^4b^2 - 7a^3b^3 + 6a^2b^4 - 3ab^5 + b^6$. *Ans.* $a^2 - ab + b^2$.

9. $a^3 - b^3 + c^3 - 3(a^2b - a^2c - ab^2 - ac^2 - b^2c + bc^2) - 6abc$. *Ans.* $a - b + c$.

10. $1 - 6x + 21x^2 - 56x^3 + 111x^4 - 174x^5 + 219x^6 - 204x^7 + 144x^8 - 64x^9$. *Ans.* $1 - 2x + 3x^2 - 4x^3$.

11. $8x^6 + 48cx^5 + 60c^2x^4 - 80c^3x^3 - 90c^4x^2 + 108c^5x - 27c^6$. *Ans.* $2x^2 + 4cx - 3c^2$.

12. $x^9 - 3x^8 + 6x^7 - 10x^6 + 12x^5 - 12x^4 + 10x^3 - 6x^2 + 3x - 1$. *Ans.* $x^3 - x^2 + x - 1$.

13. $x^9 + 6x^8 - 64x^6 - 96x^5 + 192x^4 + 512x^3 - 768x - 512$. *Ans.* $x^3 + 2x^2 - 4x - 8$.

14. $8a^3 - 12a^3b + 36a^2bc + 6a^3b^2 - 36a^2b^2c - a^3b^3 + 54ab^2c^2 + 9a^2b^3c - 27ab^3c^2 + 27b^3c^3$. *Ans.* $2a - ab + 3bc$.

271. To find the cube root of a number.

PRINCIPLES.—1. *The cube of a number contains three times as many figures as the number, or three times as many less one or two.* Thus,

$$\begin{array}{rlrl} 1^3 = & 1, & 10^3 = & 1000, \\ 3^3 = & 27, & 100^3 = & 1000000, \\ 9^3 = & 729, & 1000^3 = & 1000000000, \\ 99^3 = & 907299, & 10000^3 = & 1000000000000. \end{array}$$

Hence,

2. *If a number be separated into periods of three figures each, beginning at units' place, the number of periods will be equal to the number of figures in the cube root of that number.* Thus, there are two figures in the cube root of 405,224.

1. Let it be required to extract the cube root of 405224.

$$\begin{array}{llllr|l} & & & & & \quad\; a \quad\;\; b \\ & & & & & 405{,}224(70+4=74 \\ & & & & a^3= & 343\;000 \\ \hline 3a^2= & 70^2\times 3= & 7^2\times 300= & & 14700 & 62224=3a^2b+3ab^2+b^3 \\ 3ab= & 70\times 4\times 3= & 7\times 4\times 30= & & 840 & 62224=(3a^2+3ab+b^2)b \\ b^2= & 4^2 & & = & 16 & \\ \hline 3a^2+3ab+b^2= & & & & 15556 & \end{array}$$

Denote the tens of the root by a, and the units by b; then, since the cube of tens contains no significant figure of a lower order than thousands, a must be the greatest multiple of ten whose cube is less than 405000; that is, a must be 70. Subtracting the cube of 70 from the given number, we find the remainder to be 62224. Dividing this remainder by $3a^2$, that is, by 14700, we obtain 4 for the value of b. Adding $3ab$, that is, 840, and b^2, that is, 16, to 14700, we find the complete divisor to be 15556. Multiplying the complete divisor by 4, and subtracting the product from 62224, we find the remainder to be 0; hence, $70 + 4$, that is, 74, is the required cube root.

2. If the cube root contains more than two figures, it may be found by a similar process, as in the following example, where it

will be seen that the partial divisor at each step is obtained by multiplying the square of that part of the root already found by 3.

$$
\begin{array}{rr|l}
 & & 12{,}812{,}904(200+30+4=234 \\
 & & 8\ 000\ 000 \\ \cline{3-3}
200^2\times 3= & 120000 & 4812904 \\
200\times 30\times 3= & 18000 & 4167000 \\ \cline{3-3}
30^2= & 900 & 645904 \\ \cline{2-2}
 & 138900 & 645904 \\ \cline{2-3}
230^2\times 3= & 158700 & \\
230\times 4\times 3= & 2760 & \\
4^2= & 16 & \\ \cline{2-2}
 & 161476 &
\end{array}
$$

The work in the preceding example may be abridged as follows:

$$
\begin{array}{rr|l}
 & & 12{,}812{,}904(234 \\
 & & 8 \\ \cline{3-3}
2^2\times 300= & 1200 & 4812 \\
2\times 3\times 30= & 180 & 4167 \\ \cline{3-3}
3^2= & 9 & 645904 \\ \cline{2-2}
 & 1389 & 645904 \\ \cline{2-3}
23^2\times 300= & 158700 & \\
23\times 4\times 30= & 2760 & \\
4^2= & 16 & \\ \cline{2-2}
 & 161476 &
\end{array}
$$

RULE.

I. *Separate the given number into periods of three figures each, beginning at the units' place.*

II. *Find the greatest number whose cube is contained in the period on the left; this will be the first figure in the root. Subtract the cube of this figure from the period on the left, and to the remainder annex the next period to form a dividend.*

III. *Divide the dividend by the partial divisor, which is three hundred times the square of the part of the root already found; the quotient will be the second figure of the root.*

IV. *To the partial divisor add thirty times the product of the first and second figures of the root, also the square of the second figure; the result will be the complete divisor.*

V. *Multiply the complete divisor by the second figure of the root, and subtract the product from the dividend.*

VI. *If there are more periods to be brought down. continue the operation in the same manner as before.*

EXAMPLES.

Extract the cube root of each of the following numbers:

1. 9261. *Ans.* 21.
2. 15625. *Ans.* 25.
3. 12167. *Ans.* 23.
4. 32768. *Ans.* 32.
5. 103.823. *Ans.* 4.7.
6. 884.736. *Ans.* 9.6.
7. 12.812904. *Ans.* 2.34.
8. 8741816. *Ans.* 206.
9. 2.5. *Ans.* 1.357+.
10. .2. *Ans.* .5848+.

272. *The Higher Roots of Quantities.*—When the index of the required root contains no prime factor greater than 3, the root may be found by methods already explained. In order to show this, it will be necessary to prove the following principle:

The mn^{th} root of a quantity is equal to the m^{th} root of the n^{th} root of that quantity.

Let $$\sqrt[m]{\sqrt[n]{a}} = r \quad . \quad . \quad . \quad (1).$$

Raising both members of (1) to the m^{th} power,

$$\sqrt[n]{a} = r^m \quad . \quad . \quad . \quad (2).$$

Raising both members of (2) to the n^{th} power,

$$a = r^{mn} \quad . \quad . \quad . \quad (3).$$

Extracting the mn^{th} root of both members of (3),

$$\sqrt[mn]{a} = r \quad . \quad . \quad . \quad (4).$$

Comparing (1) and (4),

$$\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}.$$

Thus, $\sqrt[4]{16} = \sqrt{\sqrt{16}} = \sqrt{4} = 2$, $\sqrt[6]{64} = \sqrt{\sqrt{64}} = \sqrt[3]{8} = 2$, and $\sqrt[8]{a} = \sqrt{\sqrt{\sqrt{a}}}$.

EXAMPLES.

1. Extract the fourth root of $6a^2b^2 + a^4 - 4a^3b - 4ab^3 + b^4$.

$$\begin{array}{l|l}
 & a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \;\big|\; a^2 - 2ab + b^2 \\
 & a^4 \\
\hline
2a^2 - 2ab & -4a^3b + 6a^2b^2 \\
 & -4a^3b + 4a^2b^2 \\
\hline
2a^2 - 4ab + b^2 & \qquad 2a^2b^2 - 4ab^3 + b^4 \\
 & \qquad 2a^2b^2 - 4ab^3 + b^4 \\
\hline
\end{array}$$

$$\begin{array}{l|l}
 & a^2 - 2ab + b^2 \;\big|\; a - b \\
 & a^2 \\
\hline
2a - b & -2ab + b^2 \\
 & -2ab + b^2 \\
\hline
\end{array}$$

We extract the square root of the given polynomial, and thus obtain $a^2 - 2ab + b^2$; we then extract the square root of this last expression, and find the root to be $a - b$; this is, therefore, the fourth root of the proposed expression.

2. Extract the fourth root of $81x^4 - 432x^3 + 864x^2 - 768x + 256$. *Ans.* $\pm (3x - 4)$.

3. Extract the sixth root of $6a^5b + 15a^4b^2 + a^6 + 20a^3b^3 + 15a^2b^4 + b^6 + 6ab^5$. *Ans.* $\pm (a + b)$.

4. Extract the eighth root of $x^8 - 16x^7 + 112x^6 - 448x^5 + 1120x^4 - 1792x^3 + 1792x^2 - 1024x + 256$. *Ans.* $\pm (x - 2)$.

273. *Roots of Fractions.*—*The n^{th} root of a fraction is a fraction whose numerator is the n^{th} root of the given numerator, and whose denominator is the n^{th} root of the given denominator.*

Thus, $$\sqrt[n]{\frac{a^n}{b^n}} = \frac{a}{b}, \text{ for } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

274. SYNOPSIS FOR REVIEW.

CHAPTER X.

- INVOLUTION.
  - POWER . . . . .
    - *Degree.*
    - *Exponent.*
    - *Perfect.*
    - *Imperfect.*
    - *Sign.*
    - *Powers of Products.*
    - *Coefficient.*
  - POWER OF MONOMIAL. *Rule.*
  - POWER OF POLYNOMIAL. *Rule.*
  - POWER OF FRACTIONS.
- EVOLUTION.
  - ROOT . . . . . .
    - *Sign.*
    - *Index.*
  - IMAGINARY QUANTITIES.
  - ROOT OF MONOMIAL. *Rule.*
  - SQUARE ROOT.
    - *Polynomial.* Rule. Cor. 1, 2.
    - *Trinomial.*
    - *Binomial.*
  - CUBE ROOT OF POLYNOMIAL. *Rule.* Cor.
  - HIGHER ROOTS.
  - ROOTS OF FRACTIONS.

CHAPTER XI.

THEORY OF EXPONENTS.

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275. We have defined a^m, where m is a positive integer, as the product of m factors, each equal to a, and we have shown that

$$a^m \times a^n = a^{m+n};$$

and that

$$\frac{a^m}{a^n} = a^{m-n}, \quad \text{or} \quad \frac{1}{a^{n-m}},$$

according as m is greater or less than n. Hitherto an exponent has been regarded as a *positive integer*; it is, however, found convenient to use exponents which are *not* positive integers, and we now proceed to explain the meaning of such exponents.

276. As fractional exponents and negative exponents have not yet been defined, we are at liberty to define them as we please. For the sake of uniformity, we shall give such a meaning to them as will make the relation

$$a^m \times a^n = a^{m+n}$$

true, whatever m and n may be.

277. *Find the meaning of* $a^{\frac{1}{2}}$.

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a \ (\mathbf{276});$$

that is, $a^{\frac{1}{2}}$ must be such a quantity that if it be squared, the result will be a; hence, $a^{\frac{1}{2}} = \sqrt{a}$.

278. *Find the meaning of* $a^{\frac{1}{3}}$.

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a;$$

$$\therefore \quad a^{\frac{1}{3}} = \sqrt[3]{a}.$$

279. *Find the meaning of* $a^{\frac{3}{4}}$.

$$a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^3;$$

$$\therefore \quad a^{\frac{3}{4}} = \sqrt[4]{a^3}.$$

280. *Find the meaning of* $a^{\frac{1}{n}}$, *where* n *is a positive integer.*

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{ to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\text{ to } n \text{ terms}} = a^1 = a;$$

$$\therefore \quad a^{\frac{1}{n}} = \sqrt[n]{a}.$$

281. *Find the meaning of* $a^{\frac{m}{n}}$, *where* m *and* n *are positive integers.*

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \text{ to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\ldots\text{ to } n \text{ terms}} = a^m;$$

$$\therefore \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Hence, the numerator of a positive fractional exponent denotes the power to which the quantity is to be raised, and the denominator denotes the root to be extracted.

282. *Find the meaning of* a^{-2}.

$$a^3 \times a^{-2} = a^{3-2} = a.$$

Dividing both members of this equation by a^3,

$$a^{-2} = \frac{a}{a^3} = \frac{1}{a^2}.$$

283. *Find the meaning of* a^{-n}, *where* n *is any positive number, integral or fractional.*

$$a^{n+1} \times a^{-n} = a.$$

Dividing both members by a^{n+1},

$$a^{-n} = \frac{a}{a^{n+1}} = \frac{1}{a^n}.$$

Hence a^{-n} is the reciprocal of a^n.

284. It follows, from the meaning which has been given to a negative exponent, that $\frac{a^m}{a^n} = a^{m-n}$ when m is less than n, as well as when m is greater than n. For, suppose m less than n; then

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}} = a^{-(n-m)} = a^{m-n}.$$

285. General Scholium.—It thus appears that it is not *absolutely necessary* to introduce fractional and negative exponents into Algebra, since they merely supply us with a new notation for quantities which we had already the means of representing. Thus,

$$a^{\frac{2}{3}} = \sqrt[3]{a^2},\quad a^{\frac{3}{2}} = \sqrt{a^3},\quad a^{\frac{4}{2}} = \sqrt{a^4} = a^2,$$

$$a^{-3} = \frac{1}{a^3},\quad a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}},\quad a^{-\frac{4}{2}} = \frac{1}{a^{\frac{4}{2}}} = \frac{1}{a^2}.$$

If m is a positive integer, the expression a^m is read *the m^{th} power of a*, or *a m^{th} power*. But if m is not a positive integer, a^m should be read *a exponent m*. Thus, $a^{\frac{2}{3}}$ is read *a exponent two-thirds*, not "*a two-thirds power*," for there is no such power.

286. *To show that* $a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$.

Let

$$x = a^{\frac{1}{n}} \times b^{\frac{1}{n}};\ \text{then}\ x^n = \left(a^{\frac{1}{n}} \times b^{\frac{1}{n}}\right)^n = \left(a^{\frac{1}{n}}\right)^n \times \left(b^{\frac{1}{n}}\right)^n.$$

But $$\left(a^{\frac{1}{n}}\right)^n \times \left(b^{\frac{1}{n}}\right)^n = ab\ \textbf{(280)};$$

$\therefore$ $$x^n = ab;$$

whence, $$x = (ab)^{\frac{1}{n}}.$$

COR. $a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} = (ab)^{\frac{1}{n}} \times c^{\frac{1}{n}} = (abc)^{\frac{1}{n}}$. In like manner it may be shown that

$$a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} \times \dots\dots k^{\frac{1}{n}} = (abc \dots\dots k)^{\frac{1}{n}}.$$

Suppose now that there are m of these quantities $a, b, c \dots k$, and that each of them is equal to a; then the last equation becomes

$$\left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}.$$

But $$(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}} \text{ (281)};$$

$\therefore$ $$\left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}}.$$ That is,

The m^{th} power of the n^{th} root of a is equivalent to the n^{th} root of the m^{th} power of a.

287. *To show that* $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}$.

Let $x = \dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$; then $x^n = \left(\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\right)^n = \dfrac{a}{b}$ **(280-260)**;

whence, $$x = \left(\frac{a}{b}\right)^{\frac{1}{n}}.$$

288. *To show that* $\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}} = a^{\frac{1}{mn}}$.

Let $x = \left(a^{\frac{1}{m}}\right)^{\frac{1}{n}}$; then $x^n = a^{\frac{1}{m}}$; therefore $x^{mn} = a$;

whence, $$x = a^{\frac{1}{mn}}.$$

289. *To show that* $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$.

Let $x = a^{\frac{m}{n}}$; then $x^n = a^m$; and $x^{np} = a^{mp}$;

whence, $$x = a^{\frac{mp}{np}}.$$

290. *EXAMPLES.*

1. Simplify $(x^{\frac{2}{3}} \times x^{\frac{4}{7}})^{\frac{14}{13}}$. *Ans.* $x^{\frac{4}{3}}$.

2. Find the product of $a^{\frac{1}{2}}$, $a^{-\frac{1}{3}}$, $a^{-\frac{1}{4}}$, and $a^{-\frac{1}{5}}$.
Ans. $a^{-\frac{17}{60}}$.

3. Find the product of $\left(\frac{ay}{x}\right)^{\frac{1}{2}}$, $\left(\frac{bx}{y^2}\right)^{\frac{1}{3}}$, and $\left(\frac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$.
Ans. $\dfrac{y^{\frac{1}{3}}}{(bx)^{\frac{1}{6}}}$.

4. Multiply $a^{\frac{1}{2}} + b^{\frac{1}{2}} + a^{-\frac{1}{2}}b$ by $ab^{-\frac{1}{2}} - a^{\frac{1}{2}} + b^{\frac{1}{2}}$.
Ans. $a^{\frac{3}{2}}b^{-\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{-\frac{1}{2}}b^{\frac{3}{2}}$.

5. Multiply $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.
Ans. $x^{\frac{5}{2}} + x^{\frac{3}{2}}y - xy^{\frac{3}{2}} - y^{\frac{5}{2}}$.

6. Multiply $a^{\frac{7}{2}} - a^3 + a^{\frac{5}{2}} - a^2 + a^{\frac{3}{2}} - a + a^{\frac{1}{2}} - 1$ by $a^{\frac{1}{2}} + 1$.
Ans. $a^4 - 1$.

7. Multiply $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}} + a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}}$.
Ans. $a + a^{\frac{1}{3}} - 1 + a^{-\frac{1}{3}} + a^{-1}$.

8. Divide $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.
Ans. $x + y$.

9. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}a^{\frac{2}{3}} + a^{\frac{4}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}$.
Ans. $x^{\frac{2}{3}} - x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}$.

10. Divide $a^{\frac{3n}{2}} - a^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}} - a^{-\frac{n}{2}}$. *Ans.* $a^n + 1 + a^{-n}$.

11. Divide $a^{\frac{5}{2}} - a^{\frac{3}{2}}b + ab^{\frac{3}{2}} - 2a^{\frac{1}{2}}b^2 + b^{\frac{5}{2}}$ by $a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}$.
Ans. $a + a^{\frac{1}{2}}b^{\frac{1}{2}} - b$.

12. Simplify $\dfrac{a^{\frac{3}{2}} - ax^{\frac{1}{2}} + a^{\frac{1}{2}}x - x^{\frac{3}{2}}}{a^{\frac{5}{2}} - a^2x^{\frac{1}{2}} + 3a^{\frac{3}{2}}x - 3ax^{\frac{3}{2}} + a^{\frac{1}{2}}x^2 - x^{\frac{5}{2}}}$.

Ans. $\dfrac{a + x}{a^2 + 3ax + x^2}$.

13. Extract the square root of $\dfrac{y^2}{x} + \dfrac{x^2}{4y} + \dfrac{2y^{\frac{3}{2}} - x^{\frac{3}{2}}}{(xy)^{\frac{1}{4}}}$.

Ans. $\dfrac{y}{x^{\frac{1}{2}}} + x^{\frac{1}{4}}y^{\frac{1}{4}} - \dfrac{x}{2y^{\frac{1}{2}}}$.

14. Extract the square root of $4a - 12a^{\frac{1}{2}}b^{\frac{1}{3}} + 9b^{\frac{2}{3}} + 16a^{\frac{1}{2}}c^{\frac{1}{4}} - 24b^{\frac{1}{3}}c^{\frac{1}{4}} + 16c^{\frac{1}{2}}$.

Ans. $2a^{\frac{1}{2}} - 3b^{\frac{1}{3}} + 4c^{\frac{1}{4}}$.

15. If $a^b = b^a$, show that $\left(\dfrac{a}{b}\right)^{\frac{a}{b}} = a^{\frac{a}{b}-1}$; and if $a = 2b$, show that $b = 2$.

291. SYNOPSIS FOR REVIEW.

CHAPTER XI.
THEORY OF EXPONENTS.

- *Basis of theory* (**276**).
- *Meaning of* $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{3}{4}}$.
- *Meaning of* $a^{\frac{1}{n}}$, $a^{\frac{m}{n}}$.
- *Meaning of* a^{-2}, a^{-n}.
- *General Scholium.*
- *Show that* $a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$.
- *Show that* $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}$.
- *Show that* $\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}} = a^{\frac{1}{mn}}$.
- *Show that* $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$.

CHAPTER XII.

RADICAL QUANTITIES.

DEFINITIONS.

292. ***A Simple Radical Quantity*** is an expression of the form of $a\sqrt[n]{b}$, or $ab^{\frac{1}{n}}$. Thus, $2\sqrt{9}$, $3\sqrt[4]{8}$, $b\sqrt[5]{a}$, and $[(a+b)^2]^{\frac{1}{3}}c$ are simple radical quantities.

293. ***The Radical Factor*** is the indicated root, and the ***Coefficient*** of the radical factor is the quantity affixed to the radical sign. Thus, in the expression $2\sqrt{9}$, the radical factor is $\sqrt{9}$ and 2 is its coefficient; and in the expression $[(a+b)^2]^{\frac{1}{3}}c$, the radical factor is $[(a+b)^2]^{\frac{1}{3}}$ and c is its coefficient.

If the coefficient of a radical factor is 1, it is usually omitted. Thus, $\sqrt{3}$ is equivalent to $1\sqrt{3}$.

294. ***The Degree*** of a simple radical quantity is denoted by the index of the radical sign, or by the denominator of the fractional exponent. Thus, $b\sqrt{a}$ and $a(b+c)^{\frac{3}{2}}$ are of the second degree; $b\sqrt[3]{a}$ and $a(b+c)^{\frac{2}{3}}$ are of the third degree; $b\sqrt[4]{a}$ and $a(b+c)^{\frac{3}{4}}$ are of the fourth degree; and so on.

295. Two or more simple radical quantities are said to be ***Similar*** if their radical factors are identical. Thus, $2\sqrt[3]{3}$ and $4\sqrt[3]{3}$ are similar.

296. A simple radical quantity is said to be in its ***Simplest Form*** when the quantity under the radical sign is *entire* and

does not contain a factor which is a perfect power corresponding to the degree of the indicated root. Thus, $3\sqrt{5}$ is in its simplest form.

297. ***A Rational Quantity*** is one which may be exactly expressed without using the radical sign or a fractional exponent. Thus, 5, -3, 4^2, and $a+b$ are rational.

Any rational quantity may be expressed under the form of a radical quantity. Thus, $5=\sqrt{25}$, $-3=-\sqrt{9}$, $4^2=\sqrt{4^4}$, and $a+b=\sqrt{(a+b)^2}$.

298. ***An Irrational Quantity*** is one which cannot be exactly expressed without using the radical sign or a fractional exponent. Thus, $3\sqrt{8}$, $2\sqrt[3]{9}$, and $5\sqrt[4]{3}$ are irrational.

Irrational quantities are sometimes called *surd quantities*, or simply *surds*.

REDUCTION OF SIMPLE RADICAL QUANTITIES.

299. ***The Reduction*** of radical quantities consists in changing their forms without altering their values.

300. To reduce a rational quantity to a radical quantity of the n^{th} degree.

$$3=\sqrt{9}=\sqrt[3]{27}=\sqrt[4]{81}=\sqrt[n]{3^n};$$

$$a=\sqrt{a^2}=\sqrt[3]{a^3}=\sqrt[4]{a^4}=\sqrt[n]{a^n};$$

and

$$a+x=\sqrt{(a+x)^2}=\sqrt[n]{(a+x)^n}.$$

RULE.

Raise the given quantity to the n^{th} power and indicate the n^{th} root of the result.

EXAMPLES.

1. Reduce $2a^2$ to a radical quantity of the third degree.
Ans. $\sqrt[3]{8a^6}$.

2. Reduce $a+x$ to a radical quantity of the fifth degree.
Ans. $\sqrt[5]{(a+x)^5}$.

3. Reduce $\frac{x}{y}$ to a radical quantity of the sixth degree.

Ans. $\sqrt[6]{\frac{x^6}{y^6}}$.

4. Reduce $-5a^2b$ to a radical quantity of the third degree.

Ans. $\sqrt[3]{-125a^6b^3}$.

5. Reduce $-(x+y)$ to a radical quantity of the fourth degree.

Ans. $-\sqrt[4]{(x+y)^4}$.

6. Reduce $(a-b)^2$ to a radical quantity of the n^{th} degree.

Ans. $\sqrt[n]{(a-b)^{2n}}$.

301. To introduce the coefficient of the radical factor under the radical sign.

$$4\sqrt{2} = \sqrt{16} \times \sqrt{2}\ \mathbf{(297)} = \sqrt{32}\ \mathbf{(286)};$$

$$a\sqrt{x} = \sqrt{a^2} \times \sqrt{x} = \sqrt{a^2x};$$

$$ay^{\frac{3}{2}} = (a^2y^3)^{\frac{1}{2}};$$

$$x\sqrt[3]{2a-x} = \sqrt[3]{x^3} \times \sqrt[3]{2a-x} = \sqrt[3]{2ax^3 - x^4}.$$

Hence, denoting the degree of the radical quantity by n, we have the following

RULE.

Multiply the quantity under the radical sign by the n^{th} power of the coefficient and indicate the n^{th} root of the product.

COR.—In a similar manner any *factor* of the coefficient may be placed under the radical sign. Thus, $3 \times 2\sqrt{5} = 3\sqrt{20}$.

EXAMPLES.

Introduce the coefficient of the radical factor, in each of the following expressions, under the radical sign:

1. $6\sqrt{5}$. *Ans.* $\sqrt{180}$.

2. $3\sqrt[4]{3}$. *Ans.* $\sqrt[4]{243}$.

3. $(a+b)\sqrt[3]{a+b}$. *Ans.* $\sqrt[3]{(a+b)^4}$.

4. $a\sqrt{\dfrac{b}{a}}$. *Ans.* $\sqrt{ab}$.

5. $(a-b)\sqrt{\dfrac{5ab^3}{a^2-2ab+b^2}}$. *Ans.* $\sqrt{5ab^3}$.

6. $\dfrac{a}{c}\sqrt{\dfrac{c}{a}+\dfrac{a}{c}}$. *Ans.* $\sqrt{\dfrac{a}{c}+\dfrac{a^3}{c^3}}$.

7. $5a\sqrt[n]{bc}$. *Ans.* $\sqrt[n]{5^n a^n bc}$.

8. $5x\sqrt[n]{25x^{-2}}$. *Ans.* $\sqrt[n]{5^{n+2}x^{n-2}}$.

9. $3x^2(x-y)^{\frac{1}{3}}$. *Ans.* $(27x^7-27x^6y)^{\frac{1}{3}}$.

10. $x\left(\dfrac{1}{x}-\dfrac{a}{x^3}+\dfrac{a^2}{x^5}\right)^{\frac{1}{n}}$ *Ans.* $(x^{n-1}-ax^{n-3}+a^2x^{n-5})^{\frac{1}{n}}$.

302. To remove a factor from under the radical sign to the coefficient.

The reduction is performed by reversing the process of Art. **301**.

Thus, $2\sqrt{8}=2\sqrt{4}\times\sqrt{2}=2\times2\sqrt{2}$, and $a\sqrt[n]{b^nc}=ab\sqrt[n]{c}$.

Hence, denoting the degree of the radical quantity by n, we have the following

RULE.

Divide the quantity under the radical sign by the factor to be removed; and to the indicated n^{th} root of the quotient prefix, as a coefficient, the product of the given coefficient and the n^{th} root of the factor to be removed.

EXAMPLES.

1. Reduce $\sqrt{20}$ to such a form that the factor 4 shall not occur under the radical sign. *Ans.* $2\sqrt{5}$.

2. Reduce $\sqrt[3]{24}$ to such a form that the factor 8 shall not occur under the radical sign. *Ans.* $2\sqrt[3]{3}$.

3. Reduce $3\sqrt{75}$ to such a form that the factor 25 shall not occur under the radical sign. *Ans.* $15\sqrt{3}$.

4. Reduce $(a - b)\sqrt{(a + b)^2 c}$ to such a form that the factor $(a + b)^2$ shall not occur under the radical sign.

Ans. $(a^2 - b^2)\sqrt{c}$.

5. Reduce $a(b + c)\sqrt[n]{b^n c}$ to such a form that the factor b^n shall not occur under the radical sign.

303. To reduce the indicated root of a fraction to an equivalent expression in which the quantity under the radical sign shall be entire.

$$\sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9} \times 6} = \sqrt{\frac{1}{9}} \times \sqrt{6}\ (\mathbf{286}) = \frac{1}{3}\sqrt{6};$$

$$\sqrt[3]{\frac{3}{5}} = \sqrt[3]{\frac{3 \times 25}{5 \times 25}} = \sqrt[3]{\frac{75}{125}} = \sqrt[3]{\frac{1}{125} \times 75} = \sqrt[3]{\frac{1}{125}} \times \sqrt[3]{75}$$
$$= \frac{1}{5}\sqrt[3]{75};$$

$$\sqrt[3]{\frac{12}{32}} = \sqrt[3]{\frac{12 \div 4}{32 \div 4}} = \sqrt[3]{\frac{3}{8}} = \sqrt[3]{\frac{1}{8} \times 3} = \sqrt[3]{\frac{1}{8}} \times \sqrt[3]{3} = \frac{1}{2}\sqrt[3]{3};$$

$$\sqrt[n]{\frac{a}{b}} = \sqrt[n]{\frac{a \times b^{n-1}}{b \times b^{n-1}}} = \sqrt[n]{\frac{ab^{n-1}}{b^n}} = \sqrt[n]{\frac{1}{b^n} \times ab^{n-1}} = \sqrt[n]{\frac{1}{b^n}} \times \sqrt[n]{ab^{n-1}}$$
$$= \frac{1}{b}\sqrt[n]{ab^{n-1}}.$$

Hence, denoting the degree of the radical quantity by n, we have the following

RULES.

I. *If the fraction under the radical sign has a denominator which is a perfect n^{th} power, prefix to the indicated n^{th} root of the numerator the reciprocal of the n^{th} root of the denominator.*

II. *If the fraction under the radical sign has a denominator which is not a perfect n^{th} power, multiply or divide both of its*

terms by such a quantity as will reduce it to one whose denominator is a perfect n^{th} power; then substitute this fraction for the given one and proceed as directed in I.

COR.—If the given radical quantity has a coefficient, the result obtained by the rule must be multiplied by it, in order to obtain an expression equivalent to the given one. Thus,

$$2\sqrt{\frac{6}{9}} = 2 \times \frac{1}{3}\sqrt{6} = \frac{2}{3}\sqrt{6}.$$

EXAMPLES.

Reduce each of the following expressions to another in which the quantity under the radical sign shall be entire:

1. $\sqrt{\frac{2}{3}}$. *Ans.* $\frac{1}{3}\sqrt{6}$.

2. $\frac{5}{2}\sqrt[3]{\frac{24}{5}}$. *Ans.* $\frac{1}{2}\sqrt[3]{600}$.

3. $2\sqrt{\frac{3}{4}}$. *Ans.* $\sqrt{3}$.

4. $\frac{3}{5}\sqrt{1\frac{2}{3}}$. *Ans.* $\frac{1}{5}\sqrt{15}$.

5. $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}}$. *Ans.* $\frac{1}{3}\sqrt[3]{18a}$.

6. $m(a+x)\sqrt[n]{\frac{a-x}{a+x}}$. *Ans.* $m\sqrt[n]{(a-x)(a+x)^{n-1}}$.

304. To reduce a simple radical quantity to its simplest form.

1. Reduce $3\sqrt{8}$ to its simplest form.

The largest perfect square which is a factor of 8 is 4. Removing this factor from the radical sign to the coefficient, we have

$$3\sqrt{8} = 6\sqrt{2}\ \textbf{(302)}.$$

2. Reduce $5\sqrt[3]{48a^6x^4}$ to its simplest form.

The largest perfect cube which is a factor of $48a^6x^4$ is $8a^6x^3$. Removing this factor from the radical sign to the coefficient, we have

$$5\sqrt[3]{48a^6x^4} = 10a^2x\sqrt[3]{6x}.$$

3. Reduce $4\sqrt[3]{\frac{8}{9}}$ to its simplest form.

$$4\sqrt[3]{\frac{8}{9}} = 4\sqrt[3]{\frac{24}{27}} = \frac{4}{3}\sqrt[3]{24}\ \textbf{(303)} = \frac{4}{3} \times 2\sqrt[3]{3}\ \textbf{(302)} = \frac{8}{3}\sqrt[3]{3}.$$

Hence, denoting the degree of the radical quantity by n, we have the following

RULES.

I. *If the quantity under the radical sign is entire, resolve it into two factors, one of which is the greatest n^{th} power contained in that quantity; then remove this factor from the radical sign to the coefficient.*

II. *If the given radical quantity contains the indicated root of a fraction, reduce it to such a form that the quantity under the radical sign shall be entire, and proceed with the result as directed in* I.

EXAMPLES.

Reduce each of the following radical quantities to its simplest form:

1. $\sqrt{25a^3b}$. *Ans.* $5a\sqrt{ab}$.
2. $\sqrt{27a^4b^3c^2x}$. *Ans.* $3a^2bc\sqrt{3bx}$.
3. $\sqrt{192a^5b^8c^7}$. *Ans.* $8a^2b^4c^3\sqrt{3ac}$.
4. $\sqrt[3]{108a^3b}$. *Ans.* $3a\sqrt[3]{4b}$.
5. $6\sqrt[3]{ax^3 + bx^6}$. *Ans.* $6x\sqrt[3]{a + bx^3}$.
6. $7\sqrt[4]{625a^4b^4c}$. *Ans.* $35ab\sqrt[4]{c}$.

7. $3\sqrt[m]{a^{m+n}b}$. *Ans.* $3a\sqrt[m]{a^n b}$.

8. $\sqrt[m]{(a+x)^m b^n}$. *Ans.* $(a+x)\sqrt[m]{b^n}$.

9. $\frac{1}{2}\sqrt{\frac{3}{7}}$. *Ans.* $\frac{1}{14}\sqrt{21}$.

10. $6\sqrt[3]{\frac{2}{3}}$. *Ans.* $2\sqrt[3]{18}$.

11. $\frac{a}{b}\sqrt{\frac{c^2}{d}}$. *Ans.* $\frac{ac}{bd}\sqrt{d}$.

12. $2\sqrt{\frac{ab^2}{4(a+x)}}$. *Ans.* $\frac{b}{a+x}\sqrt{(a+x)a}$.

13. $\left(\frac{a^4b^8}{xy}\right)^{\frac{1}{5}}$. *Ans.* $\frac{b}{xy}(a^4b^3x^4y^4)^{\frac{1}{5}}$.

14. $\left(\frac{a^{-1}b^{-2}}{x}\right)^{\frac{1}{3}}$. *Ans.* $\frac{1}{abx}(a^2bx^2)^{\frac{1}{3}}$.

15. $\sqrt[3]{\frac{5(a^3+a^4b)}{b^7}}$. *Ans.* $\frac{a}{b^3}\sqrt[3]{5(1+ab)b^2}$.

16. $(a+b)\sqrt{\frac{a-b}{a+b}}$. *Ans.* $\sqrt{a^2-b^2}$.

17. $(a-b)\sqrt{\frac{a^2c+2abc+b^2c}{m+n}}$. *Ans.* $\frac{a^2-b^2}{m+n}\sqrt{c(m+n)}$.

18. $\left(\frac{a-x}{a+x}\right)^{\frac{1}{n}}$. *Ans.* $\frac{1}{a+x}[(a-x)(a+x)^{n-1}]^{\frac{1}{n}}$.

305. To reduce a radical quantity of the form of $\sqrt[mn]{a^n}$ to another of a lower degree.

$$\sqrt[4]{9} = \sqrt{\sqrt{9}}\ (272) = \sqrt{3};$$

$$\sqrt[6]{8} = \sqrt{\sqrt[3]{8}} = \sqrt{2};$$

$$\sqrt[mn]{a^n} = \sqrt[m]{\sqrt[n]{a^n}} = \sqrt[m]{a}.$$

Hence, denoting the factors of the index by m and n, we have the following

RULE.

Extract the n^{th} root of the quantity under the radical sign, and indicate the m^{th} root of the result.

EXAMPLES.

1. Reduce $\sqrt[6]{4a^2}$ to a radical quantity of the third degree.
Ans. $\sqrt[3]{2a}$.

2. Reduce $\sqrt[4]{64a^2b^2}$ to a radical quantity of the second degree.
Ans. $\sqrt{8ab}$.

3. Reduce $\sqrt[8]{16a^4b^4c^{12}}$ to a radical quantity of the second degree.
Ans. $\sqrt{2abc^3}$.

4. Reduce $\sqrt[6]{25a^2b^2c^4}$ to a radical quantity of the third degree.
Ans. $\sqrt[3]{5abc^2}$.

5. Reduce $\sqrt[9]{a^6b^3c^6}$ to a radical quantity of the third degree.
Ans. $\sqrt[3]{a^2bc^2}$.

6. Reduce $\sqrt[10]{a^2b^4c^6}$ to a radical quantity of the fifth degree.
Ans. $\sqrt[5]{ab^2c^3}$.

306. **To reduce a simple radical quantity to another of a higher or lower degree.**

$$\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{2}{4}} = a^{\frac{3}{6}} = a^{\frac{4}{8}} = a^{\frac{n}{2n}} = \sqrt[2n]{a^n};$$

$$\sqrt[12n]{a^{4n}} = a^{\frac{4n}{12n}} = a^{\frac{4}{12}} = a^{\frac{3}{9}} = a^{\frac{2}{6}} = a^{\frac{1}{3}} = \sqrt[3]{a}.$$

RULE.

Express the given radical quantity by means of a fractional exponent; then substitute for this exponent any equivalent fraction having a denominator greater or less than that of the given fractional exponent.

COR. 1.—If equal factors be introduced into the index of the root and the exponent of the quantity under the radical sign, the result will be equal to the given radical quantity. Thus, $\sqrt[3]{a} = \sqrt[12]{a^4} = \sqrt[3n]{a^n}$.

COR. 2.—Conversely, if equal factors be canceled in the index of the root and the exponent of the quantity under the radical sign, the result will be equal to the given radical quantity. Thus, $\sqrt[3n]{a^n} = \sqrt[3]{a}$.

EXAMPLES.

1. Reduce $\sqrt{a}$ to a radical quantity of the 12th degree.

Ans. $\sqrt[12]{a^6}$.

2. Reduce $\sqrt[m]{a}$ to a radical quantity of the mn^{th} degree.

Ans. $\sqrt[mn]{a^n}$.

3. Reduce $\sqrt[4]{a-b}$ to a radical quantity of the 20th degree.

Ans. $\sqrt[20]{(a-b)^5}$.

4. Reduce $\sqrt[5]{(a+b)^2}$ to a radical quantity of the 10th degree.

Ans. $\sqrt[10]{(a+b)^4}$.

5. Reduce $\sqrt[15]{(a-b)^{10}}$ to a radical quantity of the 3d degree.

Ans. $\sqrt[3]{(a-b)^2}$.

307. To reduce simple radical quantities having unequal indices to equivalent ones having equal indices.

1. Reduce $\sqrt{a}$ and $\sqrt[3]{a}$ to equivalent expressions having equal indices.

$$\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3};$$

and
$$\sqrt[3]{a} = a^{\frac{1}{3}} = a^{\frac{2}{6}} = \sqrt[6]{a^2}.$$

2. Reduce $\sqrt[m]{a}$ and $\sqrt[n]{a}$ to equivalent expressions having equal indices.

$$\sqrt[m]{a} = a^{\frac{1}{m}} = a^{\frac{n}{mn}} = \sqrt[mn]{a^n};$$

and
$$\sqrt[n]{a} = a^{\frac{1}{n}} = a^{\frac{m}{mn}} = \sqrt[mn]{a^m}.$$

RULE.

Express the indicated roots by means of fractional exponents, and reduce the expressions thus obtained to equivalent ones, in which the fractional exponents shall have equal denominators.

EXAMPLES.

1. Reduce $\sqrt[12]{2}$ and $3\sqrt[4]{3}$ to equivalent expressions having equal indices. *Ans.* $\sqrt[12]{2}$ and $3\sqrt[12]{27}$.

2. Reduce $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$, and $\sqrt[5]{5}$ to equivalent expressions having equal indices. *Ans.* $\sqrt[60]{2^{30}}$, $\sqrt[60]{3^{20}}$, $\sqrt[60]{4^{15}}$, $\sqrt[60]{5^{12}}$.

3. Reduce $\sqrt{2}$, $\sqrt[4]{3}$, $\sqrt[6]{5}$, and $\sqrt[8]{7}$, to equivalent expressions having equal indices.

Ans. $\sqrt[24]{4096}$, $\sqrt[24]{729}$, $\sqrt[24]{625}$, $\sqrt[24]{343}$.

4. Reduce $3^{\frac{2}{3}}$, $2^{\frac{3}{4}}$, and $5^{\frac{1}{2}}$ to equivalent expressions having equal indices. *Ans.* $3^{\frac{8}{12}}$, $2^{\frac{9}{12}}$, $5^{\frac{6}{12}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to equivalent expressions having equal indices. *Ans.* $\sqrt[mn]{a^m}$ and $\sqrt[mn]{b^n}$.

6. Reduce $\sqrt{ax}$, $\sqrt[m]{xy}$, and $\sqrt[n]{cx}$ to equivalent expressions having equal indices.

Ans. $\sqrt[2mn]{a^{mn}x^{mn}}$, $\sqrt[2mn]{x^{2n}y^{2n}}$, $\sqrt[2mn]{c^{2m}x^{2m}}$.

7. Reduce $\sqrt[10]{a^5}$, $\sqrt[6]{b^3}$, $\sqrt[8]{c^4}$, $\sqrt[2n]{d^n}$, and $\sqrt[n]{e^{\frac{n}{2}}}$ to equivalent expressions having equal indices.

Ans. $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, $\sqrt{d}$, $\sqrt{e}$.

COMBINATIONS OF RADICAL QUANTITIES.

308. To find the sum of two or more simple radical quantities.

1. Find the sum of $6\sqrt{2}$ and $8\sqrt{2}$.

$$6\sqrt{2}+8\sqrt{2}=(6+8)\sqrt{2}=14\sqrt{2}.$$

2. Find the sum of $2\sqrt[3]{24}$ and $3\sqrt[3]{192}$.

$$2\sqrt[3]{24}=2\sqrt[3]{8\times 3}=4\sqrt[3]{3},$$

and $$3\sqrt[3]{192}=3\sqrt[3]{64\times 3}=12\sqrt[3]{3};$$

$\therefore$ $$2\sqrt[3]{24}+3\sqrt[3]{192}=16\sqrt[3]{3}.$$

3. Find the sum of $\sqrt{x^3}$, $\sqrt{4x^3}$, and $\sqrt{a^2x}$.

$$\sqrt{x^3}=x\sqrt{x},$$

$$\sqrt{4x^3}=2x\sqrt{x},$$

and $\sqrt{a^2x}=a\sqrt{x};$

$\therefore$ $\sqrt{x^3}+\sqrt{4x^3}+\sqrt{a^2x}=x\sqrt{x}+2x\sqrt{x}+a\sqrt{x}=(3x+a)\sqrt{x}.$

4. Find the sum of $2\sqrt[3]{108}$ and $5\sqrt[3]{24}$.

$$2\sqrt[3]{108}=2\sqrt[3]{27\times 4}=6\sqrt[3]{4},$$

and $$5\sqrt[3]{24}=5\sqrt[3]{8\times 3}=10\sqrt[3]{3};$$

$\therefore$ $$2\sqrt[3]{108}+5\sqrt[3]{24}=6\sqrt[3]{4}+10\sqrt[3]{3}.$$

In this example the radical quantities cannot be made similar; hence the addition can only be indicated.

5. Find the sum of $2\sqrt[4]{36}$ and $3\sqrt{6}$.

$3\sqrt{6}=3\sqrt[4]{36}$ (**306,** Cor. 1).

$\therefore$ $2\sqrt[4]{36}+3\sqrt{6}=2\sqrt[4]{36}+3\sqrt[4]{36}=5\sqrt[4]{36}=5\sqrt{6}$ (**305**).

RULES.

I. *If the given radical quantities are similar, prefix the sum of the coefficients to the common radical factor.*

II. *If the given radical quantities are of the same degree, but not similar, reduce them, if possible, to equivalent similar ones by the rule of Art.* **304,** *and proceed with the results as directed in* I. *If they cannot be so reduced, indicate their sum.*

III. *If the given radical quantities are of different degrees, reduce them to equivalent ones of the same degree, and proceed with the results as directed in* II.

EXAMPLES.

1. Find the sum of $7\sqrt{10}$ and $2\sqrt{90}$. *Ans.* $13\sqrt{10}$.

2. Find the sum of $\sqrt[3]{500}$ and $\sqrt[3]{256}$. *Ans.* $9\sqrt[3]{4}$.

3. Find the sum of $4\sqrt[3]{500}$ and $3\sqrt[3]{108}$. *Ans.* $29\sqrt[3]{4}$.

4. Find the sum of $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{2}{9}}$, and $\sqrt{\frac{1}{18}}$. *Ans.* $\sqrt{2}$.

5. Find the sum of $\frac{3}{5}\sqrt{\frac{2}{3}}$, $\frac{3}{4}\sqrt{\frac{25}{6}}$, and $\frac{7}{8}\sqrt{\frac{96}{25}}$.

Ans. $\frac{61}{40}\sqrt{6}$.

6. Find the sum of $\sqrt{a^2x}$ and $\sqrt{b^2x}$. *Ans.* $(a+b)\sqrt{x}$.

7. Find the sum of $\sqrt{24}$, $2\sqrt{72}$, and $a\sqrt{6x^2}$.

Ans. $2\left(\sqrt{6}+6\sqrt{2}\right)+ax\sqrt{6}$.

8. Find the sum of $\sqrt[3]{\frac{27a^5x}{2b}}$ and $\sqrt[3]{\frac{a^2x}{2b}}$.

Ans. $\frac{3a+1}{2b}\sqrt[3]{4a^2b^2x}$.

9. Find the sum of $\sqrt{(1+a)^{-1}}$, $\sqrt{a^2(1+a)^{-1}}$, and $a\sqrt{(1+a)(1+a)^2}$. *Ans.* $(a^2+a+1)\sqrt{1+a}$.

10. Find the sum of $3\sqrt[3]{16a^4b^4c^{12}}$ and $5\sqrt{2abc^5}$.
Ans. $(3c+5c^2)\sqrt{2abc}$.

11. Find the sum of $\sqrt{2ax^2-4ax+2a}$ and $\sqrt{2ax^2+4ax+2a}$.
Ans. $2x\sqrt{2a}$.

12. Find the sum of $\sqrt[3]{54a^{m+6}b^3}$, $\sqrt[3]{16a^{m-3}b^6}$, $\sqrt[3]{2a^{4m+9}}$, and $\sqrt[3]{2c^3a^m}$. *Ans.* $\left(3a^2b+\dfrac{2b^2}{a}+a^{m+3}+c\right)\sqrt[3]{2a^m}$.

13. Find the sum of $a\sqrt[3]{b^5}$ and $c\sqrt{b^5}$.
Ans. $ab\sqrt[3]{b^2}+cb^2\sqrt{b}$.

14. Find the sum of $a\left(1+\dfrac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right)^{\frac{1}{2}}$ and $b\left(1+\dfrac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}\right)^{\frac{1}{2}}$.
Ans. $\left[\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^3\right]^{\frac{1}{2}}$.

309. To find the difference of two simple radical quantities.

1. Subtract $6\sqrt{2}$ from $8\sqrt{2}$.

$$8\sqrt{2}-6\sqrt{2}=(8-6)\sqrt{2}=2\sqrt{2}.$$

2. Subtract $2\sqrt[3]{24}$ from $3\sqrt[3]{192}$.

$$3\sqrt[3]{192}=3\sqrt[3]{64\times 3}=12\sqrt[3]{3},$$

and $$2\sqrt[3]{24}=2\sqrt[3]{8\times 3}=4\sqrt[3]{3};$$

$\therefore$ $$3\sqrt[3]{192}-2\sqrt[3]{24}=8\sqrt[3]{3}.$$

3. Subtract $\sqrt{x^3}$ from $\sqrt{4x^3}$.

$$\sqrt{4x^3}=2x\sqrt{x},$$

and $$\sqrt{x^3}=x\sqrt{x};$$

$\therefore$ $$\sqrt{4x^3}-\sqrt{x^3}=x\sqrt{x}.$$

4. Subtract $2\sqrt[3]{108}$ from $5\sqrt[3]{24}$.

$$5\sqrt[3]{24} = 5\sqrt[3]{8 \times 3} = 10\sqrt[3]{3},$$

and $$2\sqrt[3]{108} = 2\sqrt[3]{27 \times 4} = 6\sqrt[3]{4};$$

$\therefore$ $$5\sqrt[3]{24} - 2\sqrt[3]{108} = 10\sqrt[3]{3} - 6\sqrt[3]{4}.$$

In this example the radical quantities cannot be made similar; hence the subtraction can only be indicated.

5. Subtract $2\sqrt[4]{36}$ from $3\sqrt{6}$.

$3\sqrt{6} = 3\sqrt[4]{36}$ (**306**, Cor. 1);

$\therefore$ $3\sqrt{6} - 2\sqrt[4]{36} = 3\sqrt[4]{36} - 2\sqrt[4]{36} = \sqrt[4]{36} = \sqrt{6}$ (**305**).

RULES.

I. *If the given radical quantities are similar, subtract the coefficient of the radical factor in the subtrahend from that of the radical factor in the minuend, and prefix the remainder to the common radical factor.*

II. *If the given radical quantities are of the same degree, but not similar, reduce them, if possible, to equivalent similar ones by the rule of Art.* **304,** *and proceed with the results as directed in* I. *If they cannot be so reduced, indicate the subtraction.*

III. *If the given radical quantities are of different degrees, reduce them to equivalent ones of the same degree, and proceed with the results as directed in* II.

EXAMPLES.

1. Subtract $\sqrt{5a}$ from $\sqrt{45a}$. *Ans.* $2\sqrt{5a}$.

2. Subtract $\sqrt[3]{24}$ from $\sqrt[3]{192}$. *Ans.* $2\sqrt[3]{3}$.

3. Subtract $\sqrt[10]{a^{10}b^2}$ from $3a\sqrt[5]{b}$. *Ans.* $2a\sqrt[5]{b}$.

4. Subtract $3\sqrt[6]{a^3}$ from $6\sqrt[4]{a^2}$. *Ans.* $3\sqrt{a}$.

5. Subtract $(a-x)\sqrt{\dfrac{a+x}{a-x}}$ from $(a-x)\sqrt{a^2-x^2}$.

Ans. $(a-x-1)\sqrt{a^2-x^2}$.

6. Subtract $\sqrt{\dfrac{a^2b-2ab^2+b^3}{a^2+2ab+b^2}}$ from $\sqrt{\dfrac{a^2b+2ab^2+b^3}{a^2-2ab+b^2}}$.

Ans. $\dfrac{4ab}{a^2-b^2}\sqrt{b}$.

7. Subtract $\sqrt[4]{32a}$ from $2\sqrt[3]{40a}$. *Ans.* $4\sqrt[3]{5a}-2\sqrt[4]{2a}$.

8. Subtract $2\sqrt[3]{54}$ from $6\sqrt[3]{320}$. *Ans.* $24\sqrt[3]{5}-6\sqrt[3]{2}$.

310. **To find the product of two or more simple radical quantities.**

1. Multiply $6\sqrt{54}$ by $3\sqrt{2}$.

$6\sqrt{54}\times 3\sqrt{2}=6\times 3\sqrt{54}\times\sqrt{2}=18\sqrt{54\times 2}$ (**286**) $=$ $18\sqrt{108}=108\sqrt{3}$.

2. Multiply $3\sqrt{2a}$ by $2\sqrt[3]{3a}$.

$3\sqrt{2a}\times 2\sqrt[3]{3a}=3\times 2\sqrt{2a}\times\sqrt[3]{3a}=6\sqrt[6]{(2a)^3}\times\sqrt[6]{(3a)^2}$ (**306,** Cor. 1) $=6\sqrt[6]{(2a)^3(3a)^2}=6\sqrt[6]{72a^5}$.

RULES.

I. *If the given radical quantities are of the same degree, find the product of the radical factors by the principle of Art.* **286,** *and to the result prefix the product of their coefficients. Express the final result in its simplest form.*

II. *If the given radical quantities are of different degrees, reduce them to equivalent ones of the same degree, and proceed with the results as directed in* I.

Cor. 1.—If the roots are indicated by fractional exponents, the product may be found by the principles of Art. **69.**

Cor. 2.—The *product* of two or more simple radical quantities can always be reduced to a simple radical quantity.

EXAMPLES.

1. Multiply $\frac{1}{4}\sqrt{6}$ by $\frac{2}{15}\sqrt{9}$. *Ans.* $\frac{1}{30}\sqrt{54} = \frac{1}{10}\sqrt{6}$.

2. Multiply $4\sqrt[3]{\frac{2}{3}}$ by $3\sqrt[3]{\frac{5}{6}}$. *Ans.* $4\sqrt[3]{15}$.

3. Multiply $3\sqrt[3]{4}$ by $4\sqrt{3}$. *Ans.* $12\sqrt[6]{432}$.

4. Multiply $\sqrt{24a^2x}$ by $\sqrt{12x}$. *Ans.* $12ax\sqrt{2}$.

5. Multiply $b\sqrt[3]{ax}$ by $c\sqrt{xy}$. *Ans.* $bc\sqrt[6]{a^2x^5y^3}$.

6. Multiply $3\sqrt[3]{b}$ by $4\sqrt[4]{a}$. *Ans.* $12\sqrt[12]{a^3b^4}$.

7. Multiply $(a+b)^{\frac{1}{4}}$ by $(a+b)^{\frac{3}{4}}$. *Ans.* $a+b$.

8. Find the product of $\sqrt[3]{6bc^{-1}}$, $\sqrt[6]{a}$, $\sqrt[3]{3^{-1}bc^2}$, and $\sqrt[6]{a^{-1}}$.
Ans. $\sqrt[3]{2b^2c}$.

9. Multiply $(a+b)^{\frac{2}{3}}$ by $(a-b)^{\frac{2}{3}}$. *Ans.* $(a^2-b^2)^{\frac{2}{3}}$.

10. Multiply $a\sqrt[n]{x}$ by $b\sqrt[m]{y}$. *Ans.* $ab\sqrt[mn]{x^my^n}$.

311. To find the product of polynomial radical quantities.

The product of two polynomial radical quantities is found by combining the rules of Art. **310** with that of Art. **70.** If fractional exponents are used to indicate the roots, the rule of Art. **70** is sufficient.

EXAMPLES.

1. Multiply $3+\sqrt{5}$ by $2-\sqrt{5}$.

$$\begin{array}{l} 3+\sqrt{5} \\ 2-\sqrt{5} \\ \hline 6+2\sqrt{5} \\ \quad -3\sqrt{5}-5 \\ \hline 1-\sqrt{5}. \end{array}$$

Product, $1-\sqrt{5}$.

2. Multiply $x+2\sqrt{y}+3\sqrt[3]{z}$ by $x-2\sqrt{y}+3\sqrt[3]{z}$.

$$\begin{array}{l}
x+2\sqrt{y}+3\sqrt[3]{z} \\
x-2\sqrt{y}+3\sqrt[3]{z} \\
\hline
x^2+2x\sqrt{y}+3x\sqrt[3]{z} \\
\quad -2x\sqrt{y}-4y-6\sqrt{y}\times\sqrt[3]{z} \\
\qquad\qquad +3x\sqrt[3]{z}+6\sqrt{y}\times\sqrt[3]{z}+9\sqrt[3]{z^2} \\
\hline
\end{array}$$

Product, $x^2-4y+6x\sqrt[3]{z}+9\sqrt[3]{z^2}$.

3. Multiply $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

$$\begin{array}{l}
a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}} \\
a^{\frac{1}{4}}-b^{\frac{1}{2}} \\
\hline
a+a^{\frac{3}{4}}b^{\frac{1}{2}}+a^{\frac{1}{2}}b+a^{\frac{1}{4}}b^{\frac{3}{2}} \\
\quad -a^{\frac{3}{4}}b^{\frac{1}{2}}-a^{\frac{1}{2}}b-a^{\frac{1}{4}}b^{\frac{3}{2}}-b^2 \\
\hline
\end{array}$$

Product, $a-b^2$.

4. Multiply $\sqrt{8}+\sqrt{3}$ by $\sqrt{8}-\sqrt{3}$. *Ans.* 5.

5. Multiply $\sqrt{a^3}+\sqrt[3]{b^2}$ by $\sqrt{a}-2\sqrt{b^3}$.

Ans. $a^2+\sqrt[6]{a^3b^4}-2\sqrt{a^3b^3}-2\sqrt[6]{b^{13}}$.

6. Multiply $\sqrt{a}+\sqrt{b+x}$ by $\sqrt{a}-\sqrt{b+x}$.

Ans. $a-b-x$.

7. Multiply $a^{\frac{5}{2}}-2a^2b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}-8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}-32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}+2b^{\frac{1}{3}}$. *Ans.* a^3-64b^2.

8. Multiply $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ by $a^{\frac{1}{3}}-b^{\frac{1}{3}}$. *Ans.* $a-b$.

9. Multiply $x^{\frac{1}{4}}y+y^{\frac{2}{3}}$ by $x^{\frac{1}{4}}-y^{-\frac{1}{3}}$. *Ans.* $x^{\frac{1}{2}}y-y^{\frac{1}{3}}$.

312. To find the quotient of two simple radical quantities.

1. Divide $6\sqrt{54}$ by $3\sqrt{2}$.

$$\frac{6\sqrt{54}}{3\sqrt{2}}=\frac{2\sqrt{54}}{\sqrt{2}}=2\sqrt{27}\ \mathbf{(287)}\ =6\sqrt{3}.$$

2. Divide $8\sqrt{2a}$ by $2\sqrt[3]{a}$.

$$\frac{8\sqrt{2a}}{2\sqrt[3]{a}} = \frac{4\sqrt{2a}}{\sqrt[3]{a}} = \frac{4\sqrt[6]{(2a)^3}}{\sqrt[6]{a^2}} = 4\sqrt[6]{\frac{(2a)^3}{a^2}} = 4\sqrt[6]{8a}.$$

RULES.

I. *If the given radical quantities are of the same degree, divide the radical factor in the dividend by that in the divisor, by the principle of Art.* **287**, *and to the result prefix the quotient obtained by dividing the coefficient in the dividend by that in the divisor. Express the final result in its simplest form.*

II. *If the given radical quantities are of different degrees, reduce them to equivalent ones of the same degree, and proceed with the results as directed in* I.

Cor. 1.—If the roots are indicated by fractional exponents, the division may be performed by the principles of Art. **84**.

Cor. 2.—The *quotient* of two simple radical quantities can always be reduced to a simple radical quantity.

EXAMPLES.

1. Divide $8\sqrt{108}$ by $\sqrt{6}$. *Ans.* $24\sqrt{2}$.

2. Divide $\sqrt[3]{512}$ by $4\sqrt[3]{2}$. *Ans.* $\sqrt[3]{4}$.

3. Divide $12\sqrt[3]{54}$ by $3\sqrt[3]{2}$. *Ans.* 12.

4. Divide $4\sqrt[3]{12}$ by $2\sqrt{3}$. *Ans.* $\frac{2}{3}\sqrt[6]{16 \times 3^5}$.

5. Divide a by $\sqrt{a}$. *Ans.* $\sqrt{a}$.

6. Divide $\sqrt[m]{a}$ by $\sqrt[m]{b}$. *Ans.* $\frac{1}{b}\sqrt[m]{ab^{m-1}}$.

7. Divide $2ab^2c^3$ by $4\sqrt[3]{a^3bc^5d}$. *Ans.* $\frac{1}{2d}\sqrt[3]{b^5c^4d^2}$.

8. Divide $\sqrt[4]{\frac{a}{b}}$ by $\sqrt{\frac{a}{b}}$. *Ans.* $\frac{1}{a}\sqrt[4]{a^3b}$.

9. Divide $a^{\frac{2}{3}}$ by $a^{\frac{1}{2}}$. *Ans.* $a^{\frac{1}{6}}$.

10. Divide $a^{\frac{m}{n}}$ by $a^{\frac{p}{q}}$. *Ans.* $a^{\frac{mq-np}{nq}}$.

313. To find the quotient of two polynomial radical quantities.

The quotient of two polynomial radical quantities is found by combining the rules of Art. **312** with that of Art. **86**. If fractional exponents are used to indicate the roots, the rule of Art. **86** is sufficient.

EXAMPLES.

1. Divide $\sqrt[6]{a^5} - \sqrt[3]{a^2} - \sqrt{a} + \sqrt[3]{a}$ by $\sqrt[3]{a} - 1$.

$$\begin{array}{l|l}
\sqrt[6]{a^5} - \sqrt{a} - \sqrt[3]{a^2} + \sqrt[3]{a} & \sqrt[3]{a} - 1 \\
\underline{\sqrt[6]{a^5} - \sqrt{a}} & \sqrt{a} - \sqrt[3]{a} \\
\qquad - \sqrt[3]{a^2} + \sqrt[3]{a} & \\
\qquad \underline{- \sqrt[3]{a^2} + \sqrt[3]{a}} &
\end{array}$$

2. Divide $a^2 + 2a^{\frac{1}{2}}b^{\frac{2}{3}} - 4a^{\frac{3}{2}}b^{\frac{1}{2}} - 8b^{\frac{7}{6}}$ by $a^{\frac{1}{2}} - 4b^{\frac{1}{2}}$.

$$\begin{array}{l|l}
a^2 - 4a^{\frac{3}{2}}b^{\frac{1}{2}} + 2a^{\frac{1}{2}}b^{\frac{2}{3}} - 8b^{\frac{7}{6}} & a^{\frac{1}{2}} - 4b^{\frac{1}{2}} \\
\underline{a^2 - 4a^{\frac{3}{2}}b^{\frac{1}{2}}} & a^{\frac{3}{2}} + 2b^{\frac{2}{3}} \\
\qquad\qquad 2a^{\frac{1}{2}}b^{\frac{2}{3}} - 8b^{\frac{7}{6}} & \\
\qquad\qquad \underline{2a^{\frac{1}{2}}b^{\frac{2}{3}} - 8b^{\frac{7}{6}}} &
\end{array}$$

3. Divide $a^2 + a\sqrt{b} - 6b$ by $a - 2\sqrt{b}$. *Ans.* $a + 3\sqrt{b}$.

4. Divide $a - 41\sqrt[5]{a} - 120$ by $\sqrt[5]{a^2} + 4\sqrt[5]{a} + 5$.

Ans. $\sqrt[5]{a^3} - 4\sqrt[5]{a^2} + 11\sqrt[5]{a} - 24$.

5. Divide $x^{\frac{5}{2}} - a^{\frac{1}{2}}x^2 - 4ax^{\frac{3}{2}} + 6a^{\frac{3}{2}}x - 2a^2x^{\frac{1}{2}}$ by $x^{\frac{3}{2}} - 4ax^{\frac{1}{2}} + 2a^{\frac{3}{2}}$.

Ans. $x - a^{\frac{1}{2}}x^{\frac{1}{2}}$.

6. Divide $x^{\frac{1}{2}} - y^{\frac{1}{2}}$ by $x^{\frac{1}{4}} - y^{\frac{1}{4}}$. *Ans.* $x^{\frac{1}{4}} + y^{\frac{1}{4}}$.

7. Divide $x^{\frac{1}{2}} - 2x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{4}} - y^{\frac{1}{4}}$. *Ans.* $x^{\frac{1}{4}} - y^{\frac{1}{4}}$.

INVOLUTION OF RADICAL QUANTITIES.

314. To find any power of the indicated n^{th} root of a quantity.

$$(\sqrt[3]{3})^2 = \sqrt[3]{3} \times \sqrt[3]{3} = \sqrt[3]{3^2}\ (\mathbf{310});$$

$$(\sqrt[3]{3})^3 = \sqrt[3]{3^2} \times \sqrt[3]{3} = \sqrt[3]{3^3} = 3;$$

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}\ (\mathbf{286}, \text{Cor.}).$$

Hence, denoting the index of the given indicated root by n, we have the following

RULE.

Raise the quantity under the radical sign to the required power, and indicate the n^{th} root of the result.

Cor. 1.—If the index of the given indicated root is equal to the exponent of the power to which that root is to be raised, the required power may be obtained by simply removing the radical sign. Thus, $(\sqrt{a})^2 = a$, $(\sqrt[3]{a})^3 = a$, and $(\sqrt[n]{a})^n = a$.

Cor. 2.—If the index of the given indicated root and the exponent of the required power contain a common factor, the result obtained by the rule may be reduced to a radical quantity of a lower degree. Thus, $(\sqrt[6]{a})^4 = \sqrt[6]{a^4} = \sqrt[3]{a^2}$ (**306**, Cor. 2).

Cor. 3.—If the root is indicated by a fractional exponent, the rule of Art. **258** is sufficient. Thus, $\left(a^{\frac{1}{2}}\right)^3 = a^{\frac{3}{2}}$, $\left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}}$.

315. To find any power of a simple radical quantity.

$$(5\sqrt[3]{3})^2 = 5\sqrt[3]{3} \times 5\sqrt[3]{3} = 5 \times 5 \times \sqrt[3]{3} \times \sqrt[3]{3} = 5^2\sqrt[3]{3^2};$$

$$(a\sqrt[n]{b})^m = a^m\sqrt[n]{b^m}.$$

Hence, denoting the exponent of the required power by m, we have the following

RULE.

Raise the given radical factor to the m^{th} power (**314**), *and to the result prefix the m^{th} power of the given coefficient.*

COR.—If the root in the given expression is indicated by a fractional exponent, the rule of Art. **258** is sufficient. Thus, $\left(2a^{\frac{1}{2}}\right)^3 = 2^3a^{\frac{3}{2}} = 8a^{\frac{3}{2}}$.

EXAMPLES.

1. Find the square of $5\sqrt[3]{a}$. *Ans.* $25\sqrt[3]{a^2}$.

2. Find the third power of $5a\sqrt[3]{x}$. *Ans.* $125a^3x$.

3. Find the square of $a^2\sqrt[3]{6}$. *Ans.* $a^4\sqrt[3]{36}$.

4. Find the 3d power of $\frac{2}{3}\sqrt{3}$. *Ans.* $\frac{8}{9}\sqrt{3}$.

5. Find the 4th power of $-\sqrt[3]{a^2}$. *Ans.* $a^2\sqrt[3]{a^2}$.

6. Find the 75th power of $x\sqrt[50]{y}$. *Ans.* $x^{75}\sqrt{y^3}$.

7. Find the square of $x\sqrt[2n]{y}$. *Ans.* $x^2\sqrt[n]{y}$.

8. Find the n^{th} power of $x\sqrt[mn]{y}$. *Ans.* $x^n\sqrt[m]{y}$.

9. Find the 4th power of $\frac{2}{3}a^{\frac{1}{3}}$. *Ans.* $\frac{16}{81}a^{\frac{4}{3}}$.

10. Find the 6th power of $a(b+c)^{\frac{1}{3}}$. *Ans.* $a^6(b^2+2bc+c^2)$.

316. To find any power of a polynomial radical quantity.

Any power of a polynomial radical quantity is found by combining the rule of Art. **315** with that of Art. **259**. If fractional exponents are used to indicate the roots, the rule of Art. **259** is sufficient.

EXAMPLES.

1. Find the square of $\sqrt{3} + a\sqrt{2}$.
Ans. $3 + 2a\sqrt{6} + 2a^2$.

2. Find the third power of $3 + \sqrt{5}$. *Ans.* $72 + 32\sqrt{5}$.

3. Find the square of $a^{\frac{1}{2}} + b^{\frac{1}{3}}$. *Ans.* $a + 2a^{\frac{1}{2}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

4. Find the 4th power of $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.
Ans. $a^2 - 4a^{\frac{3}{2}}b^{\frac{1}{2}} + 6ab - 4a^{\frac{1}{2}}b^{\frac{3}{2}} + b^2$.

EVOLUTION OF RADICAL QUANTITIES.

317. To find any root of the indicated root of a quantity.

$$\sqrt{\sqrt[3]{3^2}} = \sqrt[6]{3^2}\ (\mathbf{272}) = \sqrt[3]{3}\ (\mathbf{306},\ \text{Cor. } 2);$$

$$\sqrt[4]{\sqrt[3]{9}} = \sqrt[12]{9} = \sqrt[12]{3^2} = \sqrt[6]{3};$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

Hence, denoting the index of the given indicated root by n, and that of the required root by m, we have the following

RULES.

I. *If the quantity under the given radical sign is a perfect m^{th} power, extract the m^{th} root of it, indicate the n^{th} root of the result, and, if possible, reduce to a lower degree.*

II. *If the quantity under the given radical sign is not a perfect m^{th} power, indicate the mn^{th} root of it, and, if possible, reduce the result to a lower degree.*

Cor.—If the given indicated root is expressed by a fractional exponent, the rule of Art. **265** is sufficient. Thus,

$$\sqrt[3]{a^{\frac{3}{2}}} = a^{\frac{1}{2}}.$$

318. **To find any root of a simple radical quantity.**

$$\sqrt{25\sqrt[3]{3^2}} = 5\sqrt{\sqrt[3]{3^2}} = 5\sqrt[6]{3^2} = 5\sqrt[3]{3};$$

$$\sqrt[4]{5\sqrt[3]{9}} = \sqrt[4]{\sqrt[3]{125 \times 9}}\ \textbf{(301)} = \sqrt[12]{125 \times 9} = \sqrt[12]{1125};$$

$$\sqrt[m]{b\sqrt[n]{a}} = \sqrt[m]{\sqrt[n]{ab^n}} = \sqrt[mn]{ab^n}.$$

Hence, denoting the index of the required root by m, we have the following

RULES.

I. *If the given coefficient is a perfect m^{th} power, prefix the m^{th} root of it to the m^{th} root of the given radical factor.*

II. *If the given coefficient is not a perfect m^{th} power, introduce it under the given radical sign, and find the m^{th} root of the result* **(317)**.

Cor.—If the radical factor is expressed by means of a fractional exponent, the rule of Art. **265** is sufficient. Thus,

$$\sqrt[4]{5 \times 9^{\frac{1}{3}}} = 5^{\frac{1}{4}} \times 9^{\frac{1}{12}}.$$

EXAMPLES.

1. Find the square root of $9\sqrt[3]{3}$. *Ans.* $3\sqrt[6]{3}$.

2. Find the square root of $3\sqrt[3]{5}$. *Ans.* $\sqrt[6]{135}$.

3. Find the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$. *Ans.* $\frac{1}{3}\sqrt{3a}$.

4. Find the fourth root of $\frac{4}{9}\sqrt[3]{\frac{4}{9}}$. *Ans.* $\frac{1}{3}\sqrt[3]{12}$.

5. Find the sixth root of $a^{12}\sqrt[19]{c^4}$. *Ans.* $a^2\sqrt[57]{c^2}$.

6. Find the fourth root of $a^{16}\sqrt[3]{b^2}$. *Ans.* $a^4\sqrt[6]{b}$.

7. Find the fifteenth root of $\sqrt[3]{(a+b)^{25}}$. *Ans.* $\sqrt[9]{(a+b)^5}$.

8. Find the n^{th} root of $a^n\sqrt[n]{a^n}$. *Ans.* $a\sqrt[n]{a}$.

319. To find the square root or the cube root of a polynomial radical quantity.

The square root or the cube root of a polynomial radical quantity is found by combining the rules of Art. **318** with those of Articles **266** and **270**.

If the roots in the given expression are indicated by fractional exponents, the rules of Articles **266** and **270** are sufficient.

EXAMPLES.

1. Find the square root of $\sqrt{x} + 2\sqrt[4]{xy} + \sqrt{y}$.

$$
\begin{array}{r|l|l}
 & \sqrt{x} + 2\sqrt[4]{xy} + \sqrt{y} & \sqrt[4]{x} + \sqrt[4]{y} \\
 & \sqrt{x} & \\
\hline
2\sqrt[4]{x} + \sqrt[4]{y} & 2\sqrt[4]{xy} + \sqrt{y} & \\
 & 2\sqrt[4]{xy} + \sqrt{y} & \\
\hline
\end{array}
$$

2. Find the square root of $x^{\frac{1}{3}} - 2x^{\frac{1}{6}}y^{\frac{1}{6}} + y^{\frac{1}{3}}$.

$$
\begin{array}{r|l|l}
 & x^{\frac{1}{3}} - 2x^{\frac{1}{6}}y^{\frac{1}{6}} + y^{\frac{1}{3}} & x^{\frac{1}{6}} - y^{\frac{1}{6}} \\
 & x^{\frac{1}{3}} & \\
\hline
2x^{\frac{1}{6}} - y^{\frac{1}{6}} & -2x^{\frac{1}{6}}y^{\frac{1}{6}} + y^{\frac{1}{3}} & \\
 & -2x^{\frac{1}{6}}y^{\frac{1}{6}} + y^{\frac{1}{3}} & \\
\hline
\end{array}
$$

3. Find the square root of $4\sqrt[3]{a^2} + 12\sqrt[3]{ab} + 9\sqrt[3]{b^2}$.
Ans. $2\sqrt[3]{a} + 3\sqrt[3]{b}$.

4. Find the cube root of $a + 3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} + b$.
Ans. $\sqrt[3]{a} + \sqrt[3]{b}$.

5. Find the square root of $a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b + 2a^{\frac{1}{2}}c^{\frac{1}{2}} + 2b^{\frac{1}{2}}c^{\frac{1}{2}} + c$. *Ans.* $a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}}$.

6. Find the cube root of $8a + 36a^{\frac{2}{3}}b^{\frac{1}{3}} + 54a^{\frac{1}{3}}b^{\frac{2}{3}} + 27b$. *Ans.* $2a^{\frac{1}{3}} + 3b^{\frac{1}{3}}$.

REDUCTION OF FRACTIONS HAVING SURD DENOMINATORS TO EQUIVALENT ONES HAVING RATIONAL DENOMINATORS.

320. ***A Simple Surd*** is a surd of the form $a\sqrt[n]{b}$ or $ab^{\frac{1}{n}}$. Thus, $2\sqrt{3}$ is a simple surd.

321. ***A Polynomial Surd*** is a surd having two or more terms. Thus, $2\sqrt{3} + 3\sqrt[3]{5} - 6\sqrt[4]{7}$ is a polynomial surd.

322. To reduce a fraction whose denominator is a simple surd to an equivalent one having a rational denominator.

$$\frac{2}{5\sqrt{3}} = \frac{2\sqrt{3}}{5\sqrt{3}\times\sqrt{3}} = \frac{2\sqrt{3}}{5\sqrt{3^2}} = \frac{2\sqrt{3}}{15};$$

$$\frac{2\sqrt{5}}{3\sqrt[3]{9}} = \frac{2\sqrt{5}\times\sqrt[3]{3}}{3\sqrt[3]{9}\times\sqrt[3]{3}} = \frac{2\sqrt[6]{125}\times\sqrt[6]{9}}{3\sqrt[3]{27}} = \frac{2\sqrt[6]{1125}}{9};$$

$$\frac{\sqrt[m]{c}}{a\sqrt[n]{b}} = \frac{\sqrt[m]{c}\times\sqrt[n]{b^{n-1}}}{a\sqrt[n]{b}\times\sqrt[n]{b^{n-1}}} = \frac{\sqrt[mn]{c^n}\times\sqrt[mn]{b^{mn-m}}}{a\sqrt[n]{b^n}} = \frac{\sqrt[mn]{c^n b^{mn-m}}}{ab}.$$

Hence, denoting the degree of the denominator of the given fraction by n, we have the following

RULE.

Divide some perfect n^{th} power which is a multiple of the quantity under the radical sign in the given denominator by that quantity, and multiply both terms of the given fraction by the indicated n^{th} root of the quotient.

EXAMPLES.

Reduce each of the following fractions to an equivalent one having a rational denominator:

1. $\dfrac{2}{\sqrt{3}}$. *Ans.* $\dfrac{2\sqrt{3}}{3}$.

2. $\dfrac{2}{\sqrt[3]{3}}$. *Ans.* $\dfrac{2\sqrt[3]{9}}{3}$.

3. $\dfrac{\sqrt{3}}{2\sqrt{6}}$. *Ans.* $\dfrac{\sqrt{2}}{4}$.

4. $\dfrac{2}{5\sqrt{a^3}}$. *Ans.* $\dfrac{2\sqrt{a}}{5a^2}$.

5. $\dfrac{3a\sqrt{2x}}{2\sqrt{3x}}$. *Ans.* $\dfrac{a\sqrt{6}}{2}$.

6. $\dfrac{\sqrt{2}}{\sqrt[3]{9}}$. *Ans.* $\dfrac{\sqrt[6]{72}}{3}$.

7. $\dfrac{\sqrt{2}}{\sqrt[4]{2}}$. *Ans.* $\sqrt[4]{2}$.

8. $\dfrac{\sqrt[3]{3}}{\sqrt[4]{2}}$. *Ans.* $\dfrac{\sqrt[12]{41472}}{2}$.

9. $\dfrac{\sqrt{2}}{\sqrt[5]{3}}$. *Ans.* $\dfrac{\sqrt[10]{2^5 \times 81^2}}{3}$.

10. $\dfrac{\sqrt[5]{3}}{\sqrt{2}}$. *Ans.* $\dfrac{\sqrt[10]{288}}{2}$.

11. $\dfrac{\sqrt[4]{2}}{\sqrt[3]{3}}$. *Ans.* $\dfrac{\sqrt[12]{8 \times 9^4}}{3}$.

12. $\dfrac{a}{\sqrt{b}}$. *Ans.* $\dfrac{a\sqrt{b}}{b}$.

13. $\dfrac{a}{\sqrt[3]{b}}$. *Ans.* $\dfrac{a\sqrt[3]{b^2}}{b}$.

14. $\dfrac{a}{\sqrt[n]{b}}$. *Ans.* $\dfrac{a\sqrt[n]{b^{n-1}}}{b}$.

15. $\dfrac{a^{\frac{1}{3}}}{b^{\frac{1}{5}}}$. *Ans.* $\dfrac{a^{\frac{1}{3}}b^{\frac{4}{5}}}{b}$.

16. $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{m}}}$. *Ans.* $\dfrac{a^{\frac{1}{n}}b^{\frac{m-1}{m}}}{b}$.

17. $\dfrac{a}{b^{\frac{m}{n}}}$. *Ans.* $\dfrac{ab^{\frac{mn-m}{n}}}{b^m}$.

18. $\dfrac{b}{a\sqrt{b}}$. *Ans.* $\dfrac{\sqrt{b}}{a}$.

19. $\dfrac{b}{a\sqrt[3]{b^2}}$. *Ans.* $\dfrac{\sqrt[3]{b}}{a}$.

20. $\dfrac{b}{a\sqrt[n]{b^{n-1}}}$. *Ans.* $\dfrac{\sqrt[n]{b}}{a}$.

21. $\dfrac{b}{a\sqrt[2n]{b^2}}$. *Ans.* $\dfrac{\sqrt[n]{b^{n-1}}}{a}$.

22. $\dfrac{\sqrt{a}}{\sqrt[n]{c^{\frac{n}{2}}}}$. *Ans.* $\dfrac{\sqrt{ac}}{c}$.

323. To reduce a fraction whose denominator is a binomial surd to an equivalent one having a rational denominator.

1. Reduce $\frac{5}{2\sqrt{3}-\sqrt{2}}$ to an equivalent fraction having a rational denominator.

$$\frac{5}{2\sqrt{3}-\sqrt{2}}=\frac{5}{\sqrt{12}-\sqrt{2}}=\frac{5(\sqrt{12}+\sqrt{2})}{(\sqrt{12}-\sqrt{2})(\sqrt{12}+\sqrt{2})}$$
$$=\frac{5(\sqrt{12}+\sqrt{2})}{12-2}=\frac{\sqrt{12}+\sqrt{2}}{2}.$$

We obtain the multiplier $\sqrt{12}+\sqrt{2}$ by dividing $12-2$ by $\sqrt{12}-\sqrt{2}$.

2. Reduce $\frac{5}{2\sqrt{3}+\sqrt{2}}$ to an equivalent fraction having a rational denominator.

$$\frac{5}{2\sqrt{3}+\sqrt{2}}=\frac{5}{\sqrt{12}+\sqrt{2}}=\frac{5(\sqrt{12}-\sqrt{2})}{(\sqrt{12}+\sqrt{2})(\sqrt{12}-\sqrt{2})}$$
$$=\frac{5(\sqrt{12}-\sqrt{2})}{12-2}=\frac{\sqrt{12}-\sqrt{2}}{2}.$$

3. Reduce $\frac{c}{\sqrt[3]{a}-\sqrt[3]{b}}$ to an equivalent fraction having a rational denominator.

Dividing $a-b$ by $\sqrt[3]{a}-\sqrt[3]{b}$, we obtain $\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}$. Multiplying both terms of the given fraction by this quotient, we have

$$\frac{c(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})}{a-b}$$

4. Reduce $\frac{c}{\sqrt[3]{a}+\sqrt[3]{b}}$ to an equivalent fraction having a rational denominator.

Dividing $a+b$ by $\sqrt[3]{a}+\sqrt[3]{b}$, we obtain $\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$;

$$\frac{c}{\sqrt[3]{a}+\sqrt[3]{b}}=\frac{c(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2})}{a+b}.$$

5. Reduce $\dfrac{c}{\sqrt[4]{a}-\sqrt{b}}$ to an equivalent fraction having a rational denominator.

$$\frac{c}{\sqrt[4]{a}-\sqrt{b}}=\frac{c}{\sqrt[4]{a}-\sqrt[4]{b^2}}=\frac{c(\sqrt[4]{a^3}+\sqrt{ab}+b\sqrt[4]{a}+\sqrt{b^3})}{(\sqrt[4]{a}-\sqrt[4]{b^2})(\sqrt[4]{a^3}+\sqrt{ab}+b\sqrt[4]{a}+\sqrt{b^3})}=$$

$$\frac{c(\sqrt[4]{a^3}+\sqrt{ab}+b\sqrt[4]{a}+\sqrt{b^3})}{a-b^2}.$$

We reduce the given fraction to an equivalent one, in which the simple surds in the denominator are of the same degree, and then multiply both terms of the resulting fraction by the quotient obtained by dividing $a-b^2$ by $\sqrt[4]{a}-\sqrt[4]{b^2}$.

RULES.

I. *If the given denominator is of the form of* $\sqrt[n]{a}-\sqrt[n]{b}$, *divide the indicated difference of the quantities under its radical signs by the denominator, and multiply both terms of the given fraction by the quotient.*

II. *If the given denominator is of the form of* $\sqrt[n]{a}+\sqrt[n]{b}$, *and its indices ere even, proceed as directed in* I.

III. *If the given denominator is of the form of* $\sqrt[n]{a}+\sqrt[n]{b}$, *and its indices are odd, divide the indicated sum of the quantities under its radical signs by the denominator, and multiply both terms of the given fraction by the quotient.*

IV. *If the given denominator is not of the form of* $\sqrt[n]{a}-\sqrt[n]{b}$, *nor of the form of* $\sqrt[n]{a}+\sqrt[n]{b}$, *reduce the given fraction to an equivalent one having a denominator of one or the other of these forms* (**306**), *and proceed with the result as directed in the rule which corresponds to the form of its denominator.*

COR. 1.—If we multiply both terms of the fraction $\dfrac{c}{a \pm \sqrt{b}}$ by $a \mp \sqrt{b}$, the resulting fraction will have a rational denominator; for

$$a \pm \sqrt{b} = \sqrt{a^2} \pm \sqrt{b}, \quad \text{and} \quad a^2 - b \div (\sqrt{a^2} \pm \sqrt{b}) = a \mp \sqrt{b}.$$

COR. 2.—A fraction whose denominator is a trinomial of the form of $\sqrt{a} \pm \sqrt{b} \pm \sqrt{c}$ may be reduced to an equivalent one having a rational denominator by two multiplications. Thus,

$$\frac{d}{\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}} = \frac{d(\sqrt{a}\pm\sqrt{b}\mp\sqrt{c})}{(\sqrt{a}\pm\sqrt{b}\pm\sqrt{c})(\sqrt{a}\pm\sqrt{b}\mp\sqrt{c})} =$$

$$\frac{d(\sqrt{a}\pm\sqrt{b}\mp\sqrt{c})}{a\pm b-c\pm 2\sqrt{ab}} = \frac{d(\sqrt{a}\pm\sqrt{b}\mp\sqrt{c})(a\pm b-c\mp 2\sqrt{ab})}{(a\pm b-c\pm 2\sqrt{ab})(a\pm b-c\mp 2\sqrt{ab})} =$$

$$\frac{d(\sqrt{a}\pm\sqrt{b}\mp\sqrt{c})(a\pm b-c\mp 2\sqrt{ab})}{(a\pm b-c)^2-4ab}.$$

EXAMPLES.

Reduce each of the following fractions to an equivalent one having a rational denominator:

1. $\dfrac{2}{\sqrt{5}-\sqrt{2}}$. *Ans.* $\dfrac{2(\sqrt{5}+\sqrt{2})}{3}$.

2. $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$. *Ans.* $\dfrac{7+2\sqrt{10}}{3}$.

3. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$. *Ans.* $\dfrac{a^2+b-2a\sqrt{b}}{a^2-b}$.

4. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$. *Ans.* $\dfrac{a+b+2\sqrt{ab}}{a-b}$.

5. $\dfrac{8-5\sqrt{2}}{3(3-2\sqrt{2})}$. *Ans.* $\dfrac{4+\sqrt{2}}{3}$.

6. $\dfrac{5}{\sqrt[3]{3}-\sqrt[3]{2}}$. *Ans.* $5\,(\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4})$.

7. $\dfrac{d}{a\sqrt[4]{b}+\sqrt{c}}$. *Ans.* $\dfrac{d\,(a^3\sqrt[4]{b^3}-a^2\sqrt{bc}+ac\sqrt[4]{b}-\sqrt{c^3})}{a^4b-c^2}$.

8. $\dfrac{x}{a-\sqrt{a^2-x^2}}$. *Ans.* $\dfrac{a+\sqrt{a^2-x^2}}{x}$.

9. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$. *Ans.* $\dfrac{a+\sqrt{a^2-x^2}}{x}$.

10. $\dfrac{(x^2+x+1)^{\frac{1}{2}}-(x^2-x-1)^{\frac{1}{2}}}{(x^2+x+1)^{\frac{1}{2}}+(x^2-x-1)^{\frac{1}{2}}}$. *Ans.* $\dfrac{x^2-(x^4-x^2-2x-1)^{\frac{1}{2}}}{x+1}$.

324. ***Utility of the Two Preceding Transformations.***—The two preceding transformations enable us, in many cases, to abridge the computation of the approximate value of a numerical fraction whose denominator is a surd.

ILLUSTRATIONS.

1. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{8}}=\dfrac{\sqrt{40}+\sqrt{24}}{8}=\dfrac{2(\sqrt{10}+\sqrt{6})}{8}=\dfrac{\sqrt{10}+\sqrt{6}}{4}$.

2. $\dfrac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}=\dfrac{7(\sqrt{55}-\sqrt{15})}{8}=\dfrac{24.8024}{8}=3.1003$.

3. $\dfrac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}=\dfrac{\sqrt{42}-\sqrt{18}}{4}=\dfrac{2.2380}{4}=0.5595$.

4. $\dfrac{9+2\sqrt{10}}{18-4\sqrt{10}}=\dfrac{2(9+2\sqrt{10})^2}{164}=\dfrac{242+72\sqrt{10}}{164}=2.8639$.

5. $\dfrac{2\sqrt[3]{3}}{\sqrt[3]{5}+\sqrt[3]{6}}=\dfrac{2\sqrt[3]{3}\,(\sqrt[3]{25}-\sqrt[3]{30}+\sqrt[3]{36})}{11}=$

$\dfrac{2(\sqrt[3]{75}-\sqrt[3]{90}+\sqrt[3]{108})}{11}=.8178$.

PROPOSITIONS RELATING TO IRRATIONAL QUANTITIES.

325. *An irrational quantity cannot be expressed by a rational fraction.*

This follows from the definition of an irrational quantity (**298**).

326. *A simple quadratic surd cannot be equal to the sum of a rational quantity and a simple quadratic surd.*

For, if possible, suppose

$$\sqrt{n} = a + \sqrt{m} \quad . \quad . \quad . \quad (1),$$

in which $\sqrt{n}$ and $\sqrt{m}$ are surds.

Squaring both members of (1),

$$n = a^2 + 2a\sqrt{m} + m;$$

whence, $$\sqrt{m} = \frac{n - a^2 - m}{2a} \quad . \quad . \quad . \quad (2);$$

that is, we have $\sqrt{m}$, an irrational quantity, equal to a rational fraction, which is impossible (**325**); hence (1) cannot be true.

327. *The product of two simple quadratic surds, which are not similar, and which cannot be made similar, is irrational.*

Let $\sqrt{m}$ and $\sqrt{n}$ be two such surds; then, if possible, suppose

$$\sqrt{mn} = an \quad . \quad . \quad . \quad (1).$$

Squaring both members of (1),

$$mn = a^2n^2;$$

whence, $$\sqrt{m} = a\sqrt{n} \quad . \quad . \quad . \quad (2);$$

that is, $\sqrt{m}$ and $\sqrt{n}$ may be so reduced as to be similar. But this is contrary to the hypothesis; hence (1) cannot be true.

328. *The quotient of two simple quadratic surds, which are not similar, and which cannot be made similar, is irrational.*

Let $\sqrt{m}$ and $\sqrt{n}$ be two such surds; then, if possible, suppose

$$\sqrt{\frac{m}{n}} = an \quad . \quad . \quad . \quad (1).$$

Squaring both members of (1),

$$\frac{m}{n} = a^2n^2;$$

whence, $$\sqrt{m} = an\sqrt{n} \quad . \quad . \quad . \quad (2);$$

that is, $\sqrt{m}$ and $\sqrt{n}$ may be made similar. But this is contrary to the hypothesis; hence (1) cannot be true.

329. *The sum or difference of two simple quadratic surds, which are not similar, and which cannot be made similar, cannot be equal to a simple quadratic surd.*

Let $\sqrt{m}$ and $\sqrt{n}$ be two such surds; then, if possible, suppose

$$\sqrt{m} \pm \sqrt{n} = \sqrt{a} \quad . \quad . \quad . \quad (1).$$

Squaring both members of (1),

$$m \pm 2\sqrt{mn} + n = a;$$

whence, $$\pm\sqrt{mn} = \frac{a - m - n}{2} \quad . \quad . \quad . \quad (2).$$

But $\sqrt{mn}$ is irrational (**327**); hence we have an irrational quantity equal to a rational fraction, which is impossible (**325**); hence (1) cannot be true.

330. *In an equation, of which each member is the sum or difference of a rational quantity and a simple quadratic surd, the rational quantities of the two members are equal, and also the irrational quantities.*

Suppose $$x \pm \sqrt{y} = a \pm \sqrt{b} \quad . \quad . \quad . \quad (1),$$

in which $\sqrt{y}$ and $\sqrt{b}$ are irrational; then will $x=a$ and $\sqrt{y}=\sqrt{b}$. For suppose

$$x=a\pm n \quad . \quad . \quad . \quad (2);$$

then (1) becomes

$$a\pm n\pm\sqrt{y}=a\pm\sqrt{b};$$

whence, $\qquad n\pm\sqrt{y}=\pm\sqrt{b} \quad . \quad . \quad . \quad (3).$

But (3) is impossible **(326)**; hence (2) cannot be true. Therefore $x=a$, and consequently $\sqrt{y}=\sqrt{b}$.

331. *If* $\sqrt{a+\sqrt{b}}=x+\sqrt{y}$, *in which* $\sqrt{b}$ *and* $\sqrt{y}$ *are irrational, then* $\sqrt{a-\sqrt{b}}=x-\sqrt{y}$.

For since

$$\sqrt{a+\sqrt{b}}=x+\sqrt{y} \quad . \quad . \quad . \quad (1),$$

we have by squaring,

$$a+\sqrt{b}=x^2+2x\sqrt{y}+y \quad . \quad . \quad . \quad (2);$$

$\therefore \quad a=x^2+y \ldots (3)$, and $\sqrt{b}=2x\sqrt{y}\ldots(4)$ **(330)**.

Subtracting (4) from (3),

$$a-\sqrt{b}=x^2-2x\sqrt{y}+y \quad . \quad . \quad . \quad (5);$$

whence, $\qquad \sqrt{a-\sqrt{b}}=x-\sqrt{y} \quad . \quad . \quad . \quad (6).$

332. *If* $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, *in which* $\sqrt{b}$, $\sqrt{x}$, *and* $\sqrt{y}$ *are irrational, then* $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

For since

$$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y} \quad . \quad . \quad . \quad (1),$$

we have by squaring,

$$a+\sqrt{b}=x+2\sqrt{xy}+y \quad . \quad . \quad . \quad (2);$$

$\therefore \quad a=x+y \ldots (3),$ and $\sqrt{b}=2\sqrt{xy} \ldots (4)$ **(330).**

Subtracting (4) from (3),

$$a-\sqrt{b}=x-2\sqrt{xy}+y \quad . \quad . \quad . \quad (5);$$

whence, $\quad \sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y} \quad . \quad . \quad . \quad (6).$

SIMPLIFICATION OF COMPLEX RADICAL QUANTITIES.

333. ***A Complex Radical Quantity*** is an expression in which one radical sign includes one or more others. Thus, $\sqrt{\sqrt{8}}$, $\sqrt{9+3\sqrt[3]{5}}$, and $\sqrt[n]{a\sqrt[m]{b\sqrt[p]{c}}}$ are complex radical quantities.

334. *The complex radical quantity* $\sqrt{a \pm \sqrt{b}}$ *may be simplified if* b *is a perfect square, or if* a^2-b *is a perfect square.*

1. Suppose that b is a perfect square; then $\sqrt{a \pm \sqrt{b}}$ may be reduced to $\sqrt{a \pm c}$, in which c is the square root of b. Thus, $\sqrt{5 \pm \sqrt{9}}=\sqrt{5 \pm 3}$.

2. Suppose that $\sqrt{b}$ is a surd, and that a^2-b is a perfect square; then $\sqrt{a+\sqrt{b}}$ may be reduced to $\pm\sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}}$, and $\sqrt{a-\sqrt{b}}$ may be reduced to $\pm\sqrt{\frac{a+c}{2}} \mp \sqrt{\frac{a-c}{2}}$, in which c is the square root of a^2-b.

Assume $\quad \sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}} \quad . \quad . \quad . \quad (1).$

In this equation one or both of the terms in the first member must be irrational, because the second member is a surd;

$$\therefore \qquad \sqrt{x}-\sqrt{y}=\sqrt{a-\sqrt{b}} \quad . \; . \; . \quad (2)\ \mathbf{(331\text{-}332)}.$$

Multiplying (1) by (2),

$$x-y=\sqrt{a^2-b} \quad . \; . \; . \quad (3).$$

Squaring both members of (1),

$$x+2\sqrt{xy}+y=a+\sqrt{b} \quad . \; . \; . \quad (4).$$

In this equation $2\sqrt{xy}$ is a surd (**327**);

$$\therefore \qquad x+y=a \quad . \; . \; . \quad (5).$$

Combining (3) and (5), we find

$$\sqrt{x}=\pm\sqrt{\frac{a+\sqrt{a^2-b}}{2}} \quad . \; . \; . \quad (6),$$

and

$$\sqrt{y}=\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}} \quad . \; . \; . \quad (7).$$

$$\therefore \qquad \sqrt{a+\sqrt{b}}=\pm\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}} \; . \; . \; . \; (8),$$

and

$$\sqrt{a-\sqrt{b}}=\pm\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\mp\sqrt{\frac{a-\sqrt{a^2-b}}{2}} \; . \; . \; . \; (9).$$

But $\sqrt{a^2-b}=c$ by hypothesis;

$$\therefore \qquad \sqrt{a+\sqrt{b}}=\pm\sqrt{\frac{a+c}{2}}\pm\sqrt{\frac{a-c}{2}} \quad . \; . \; . \quad (10),$$

and

$$\sqrt{a-\sqrt{b}}=\pm\sqrt{\frac{a+c}{2}}\mp\sqrt{\frac{a-c}{2}} \quad . \; . \; . \quad (11).$$

EXAMPLES.

1. Simplify $\sqrt{3+2\sqrt{2}}$.

$$\sqrt{3+2\sqrt{2}}=\sqrt{3+\sqrt{8}};$$

$\therefore \qquad a=3, \quad b=8, \quad \text{and} \quad c=\sqrt{9-8}=1.$

Substituting in (10),

$$\sqrt{3+2\sqrt{2}}=\pm\sqrt{\frac{3+1}{2}}\pm\sqrt{\frac{3-1}{2}}=\pm\sqrt{2}\pm 1.$$

2. Simplify $\sqrt{7-2\sqrt{10}}$.

$$\sqrt{7-2\sqrt{10}}=\sqrt{7-\sqrt{40}};$$

$\therefore \qquad a=7, \quad b=40, \quad \text{and} \quad c=\sqrt{49-40}=3.$

Substituting in (11),

$$\sqrt{7-2\sqrt{10}}=\pm\sqrt{5}\mp\sqrt{2}.$$

Simplify each of the following expressions:

3. $\sqrt{11+6\sqrt{2}}$. *Ans.* $\pm 3\pm\sqrt{2}$.

4. $\sqrt{7-4\sqrt{3}}$. *Ans.* $\pm 2\mp\sqrt{3}$.

5. $\sqrt{94+42\sqrt{5}}$. *Ans.* $\pm 7\pm 3\sqrt{5}$.

6. $\sqrt{11+6\sqrt{2}}+\sqrt{7-2\sqrt{10}}$. *Ans.* $\pm 3\pm\sqrt{5}$.

7. $\sqrt{bc+2b\sqrt{bc-b^2}}$. *Ans.* $\pm b\pm\sqrt{bc-b^2}$.

8. $\sqrt{(a+b)^2-4(a-b)\sqrt{ab}}$. *Ans.* $\pm(a-b)\mp 2\sqrt{ab}$.

335. *The complex radical quantity* $\sqrt{a\sqrt{c}\pm\sqrt{b}}$, *in which* $a\sqrt{c}$ *and* $\sqrt{b}$ *are supposed to be surds, may be simplified, if* $a^2-\frac{b}{c}$ *is a perfect square.*

$$a\sqrt{c} \pm \sqrt{b} = \sqrt{c}\left(a \pm \sqrt{\frac{b}{c}}\right);$$

$$\therefore \sqrt{a\sqrt{c} \pm \sqrt{b}} = \sqrt{\sqrt{c}\left(a \pm \sqrt{\frac{b}{c}}\right)} = \sqrt[4]{c}\sqrt{a \pm \sqrt{\frac{b}{c}}} \text{ (302).}$$

This expression may now be simplified by the method of Art. **334** when $a^2 - \frac{b}{c}$ is a perfect square.

EXAMPLES.

1. Simplify $\sqrt{\sqrt{32} + \sqrt{30}}$.

$$\sqrt{32} + \sqrt{30} = \sqrt{2}\,(4 + \sqrt{15});$$

$$\therefore \qquad \sqrt{\sqrt{32} + \sqrt{30}} = \sqrt[4]{2}\sqrt{4 + \sqrt{15}}.$$

But $$\sqrt{4 + \sqrt{15}} = \pm\sqrt{\frac{5}{2}} \pm \sqrt{\frac{3}{2}};$$

$$\therefore \quad \sqrt{\sqrt{32} + \sqrt{30}} = \sqrt[4]{2}\left(\pm\sqrt{\frac{5}{2}} \pm \sqrt{\frac{3}{2}}\right) = \frac{1}{\sqrt[4]{2}}(\pm\sqrt{5} \pm \sqrt{3}).$$

Simplify each of the following expressions.

2. $\sqrt{\sqrt{27} + \sqrt{15}}$. *Ans.* $\sqrt[4]{3}\left(\pm\frac{1}{\sqrt{2}} \pm \sqrt{\frac{5}{2}}\right)$.

3. $\sqrt{5\sqrt{2} + 4\sqrt{3}}$. *Ans.* $\sqrt[4]{2}\,(\pm\sqrt{3} \pm \sqrt{2})$.

4. $\sqrt{8\sqrt{3} - 2\sqrt{45}}$. *Ans.* $\sqrt[4]{3}\,(\pm\sqrt{5} \mp \sqrt{3})$.

336. *Complex radical quantities of the form of* $\sqrt[n]{a\sqrt[m]{b}}$ *may be simplified by the rules of Art.* **318.** Thus,

$$\sqrt[3]{3\sqrt{5}} = \sqrt[3]{\sqrt{45}} = \sqrt[6]{45}.$$

337. *Complex radical quantities of the form of*

$\sqrt{a\sqrt[n]{b} \pm c\sqrt[m]{d} \pm \text{etc.}}$, *or of the form of* $\sqrt[3]{a\sqrt[n]{b} \pm c\sqrt[m]{d} \pm \text{etc.}}$,

may, in some cases, be simplified by the method of Art. **319.** Thus,

$\sqrt{\sqrt{5} + 2\sqrt[4]{15} + \sqrt{3}} = \sqrt[4]{5} + \sqrt[4]{3}.$

IMAGINARY QUANTITIES.

338. ***An Imaginary Quantity*** is one which, when in its simplest form, contains an indicated *even root* of a negative quantity. Thus, $2\sqrt{-9}$, $5\sqrt[4]{-10}$, and $(a+b)\sqrt{-1}$ are imaginary.

339. The term ***Real*** is applied to all quantities that are not imaginary. Thus, 5, -3, $\sqrt{8}$, and $\sqrt[3]{-27}$ are real.

340. Imaginary quantities are classified in the same way as other surd quantities. Thus, $2\sqrt{-3}$ is simple and of the second degree, $\sqrt{3 + 2\sqrt{-9}}$ is complex, and $8 + 3\sqrt[4]{-2} + 5\sqrt{-4} - 7\sqrt[6]{-1}$, considered as a single expression, is a compound or polynomial imaginary quantity.

An imaginary quantity usually consists of a *real* and an *imaginary* part. Thus, $2 + 3\sqrt{-1}$ consists of the real part 2 and the imaginary part $3\sqrt{-1}$. The whole expression is considered as an imaginary quantity on account of the presence of the imaginary part.

COMBINATIONS OF SIMPLE IMAGINARY QUANTITIES.

341. To find the sum of simple imaginary quantities.

EXAMPLES.

1. Find the sum of $\sqrt{-9}$ and $\sqrt{-16}$.

$$\sqrt{-9} + \sqrt{-16} = \sqrt{9(-1)} + \sqrt{16(-1)} = 3\sqrt{-1} + 4\sqrt{-1} = 7\sqrt{-1}.$$

2. Find the sum of $3\sqrt[4]{-81}$ and $2\sqrt[4]{-16}$.

$$3\sqrt[4]{-81}+2\sqrt[4]{-16}=3\sqrt[4]{81(-1)}+2\sqrt[4]{16(-1)}=$$
$$9\sqrt[4]{-1}+4\sqrt[4]{-1}=13\sqrt[4]{-1}.$$

3. Find the sum of $\sqrt{-50}$ and $\sqrt{-18}$. *Ans.* $8\sqrt{-2}$.

4. Find the sum of $\sqrt{-a}$ and $\sqrt{-b}$.

Ans. $(\sqrt{a}+\sqrt{b})\sqrt{-1}$.

342. To find the difference of two simple imaginary quantities.

EXAMPLES.

1. Subtract $2\sqrt{-4}$ from $9\sqrt{-1}$.

$$9\sqrt{-1}-2\sqrt{-4}=9\sqrt{-1}-2\sqrt{4(-1)}=9\sqrt{-1}-$$
$$4\sqrt{-1}=5\sqrt{-1}.$$

2. Subtract $2\sqrt{-3}$ from $9\sqrt{-2}$.

$$9\sqrt{-2}-2\sqrt{-3}=9\sqrt{2(-1)}-2\sqrt{3(-1)}=$$
$$9\sqrt{2}\sqrt{-1}-2\sqrt{3}\sqrt{-1}=(9\sqrt{2}-2\sqrt{3})\sqrt{-1}.$$

3. Subtract $\sqrt[4]{-16}$ from $\sqrt[4]{-81}$. *Ans.* $\sqrt[4]{-1}$.

4. Subtract $\sqrt{-b}$ from $\sqrt{-a}$.

Ans. $(\sqrt{a}-\sqrt{b})\sqrt{-1}$.

343. To find the product of two simple imaginary quantities of the second degree.

EXAMPLES.

1. Find the product of $b\sqrt{-a}$ and $c\sqrt{-a}$.

It is evident that $b\sqrt{-a}\times c\sqrt{-a}=bc(-a)$ (**314**, Cor. 1) $=-abc$. It is also evident that $b\sqrt{-a}\times c\sqrt{-a}=bc\sqrt{a^2}$;

for if this be not true, the rule for the sign of a product is not general; it therefore follows that, in this case, $\sqrt{a^2} = -a$.

But it may be said that $\sqrt{a^2} = a$, and therefore $a = -a$. This reasoning is erroneous, for it is not true that $\sqrt{a^2} = +a$ *and* $-a$ at the same time (**74**).

We are enabled to remove the ambiguity with regard to the sign of $\sqrt{a^2}$ by knowing that a^2 resulted from the involution of $-a$. If we did not know in what way a^2 was produced, that is, whether a^2 represented $(+a)^2$ or $(-a)^2$, then the sign of $\sqrt{a^2}$ would be ambiguous.

2. Find the product of $\sqrt{-a}$ and $\sqrt{-b}$.

$$\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a}\sqrt{-1},$$

and $$\sqrt{-b} = \sqrt{b(-1)} = \sqrt{b}\sqrt{-1};$$

$$\therefore \quad \sqrt{-a} \times \sqrt{-b} = \sqrt{a}\sqrt{-1} \times \sqrt{b}\sqrt{-1} = \sqrt{ab}\left(\sqrt{-1}\right)^2$$
$$= -\sqrt{ab}.$$

The ambiguity with regard to the sign of the product may therefore be removed, if we reduce each of the imaginary factors to the form of $a\sqrt{-1}$, and remember that $\sqrt{-1} \times \sqrt{-1}$ or $\left(\sqrt{-1}\right)^2$ is equal to -1.

3. Multiply $4\sqrt{-5}$ by $3\sqrt{-1}$. *Ans.* $-12\sqrt{5}$.

4. Multiply $-5\sqrt{-2}$ by $-3\sqrt{-5}$. *Ans.* $-15\sqrt{10}$.

5. Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$. *Ans.* $-ab$.

344. To find the quotient of two simple imaginary quantities of the same degree.

EXAMPLES.

1. Divide $\sqrt{-a}$ by $\sqrt{-b}$.

$$\frac{\sqrt{-a}}{\sqrt{-b}} = \frac{\sqrt{a}\sqrt{-1}}{\sqrt{b}\sqrt{-1}} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

2. Divide $-\sqrt{-a}$ by $-\sqrt{-b}$.

$$\frac{-\sqrt{-a}}{-\sqrt{-b}}=\frac{\sqrt{-a}}{\sqrt{-b}}=\sqrt{\frac{a}{b}}.$$

3. Divide $\sqrt{-a}$ by $-\sqrt{-b}$.

$$\frac{\sqrt{-a}}{-\sqrt{-b}}=\frac{\sqrt{a}\sqrt{-1}}{-\sqrt{b}\sqrt{-1}}=\frac{\sqrt{a}}{-\sqrt{b}}=-\sqrt{\frac{a}{b}}.$$

The ambiguity with regard to the sign of the quotient of two imaginary quantities is removed, therefore, by reducing each of them to the form of $a\sqrt{-1}$, and observing that $\dfrac{\sqrt{-1}}{\sqrt{-1}}=1$.

4. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. *Ans.* $\frac{3}{2}\sqrt{3}$.

5. Divide $-\sqrt{-1}$ by $-6\sqrt{-3}$. *Ans.* $\dfrac{1}{6\sqrt{3}}$.

345. To find all the powers of $\sqrt{-1}$.

$$(\sqrt{-1})^1=\sqrt{-1},$$
$$(\sqrt{-1})^2=-1,$$
$$(\sqrt{-1})^3=-1\sqrt{-1}=-\sqrt{-1},$$
$$(\sqrt{-1})^4=(-1)^2=1.$$

If we multiply these powers, in their order, by the 4th, we shall obtain the 5th, 6th, 7th, and 8th;

$$(\sqrt{-1})^5=\sqrt{-1},$$
$$(\sqrt{-1})^6=-1,$$
$$(\sqrt{-1})^7=-\sqrt{-1},$$
$$(\sqrt{-1})^8=1.$$

Therefore all the powers of $\sqrt{-1}$, arranged in order, beginning with the lowest, form a repeating cycle of the following terms: $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1.

346. MISCELLANEOUS EXAMPLES IN IMAGINARY QUANTITIES.

If the student will observe the directions given in Articles **343** and **344**, and remember that imaginary quantities are surds, he will have no difficulty in solving the following problems:

1. Multiply $4+\sqrt{-3}$ by $\sqrt{-5}$.
   Ans. $4\sqrt{-5}-\sqrt{15}$.

2. Multiply $3+\sqrt{-2}$ by $2-\sqrt{-4}$.
   Ans. $6+2\sqrt{-2}-3\sqrt{-4}+\sqrt{8}$.

3. Multiply $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* 2.

4. Multiply $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$. *Ans.* a^2+b^2.

5. Divide $(\sqrt{-1})^4$ by $\sqrt{-1}$. *Ans.* $-\sqrt{-1}$.

6. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$. *Ans.* $1+\sqrt{-2}$.

7. Reduce $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$ to an equivalent fraction having a rational denominator. *Ans.* $\sqrt{-1}$.

8. Simplify $\sqrt{7+30\sqrt{-2}}$. *Ans.* $\pm 5 \pm 3\sqrt{-2}$.

9. Simplify $\sqrt{31+12\sqrt{-5}}+\sqrt{-1+4\sqrt{-5}}$.
   Ans. $\pm 8 \pm 2\sqrt{-5}$.

10. Simplify $\sqrt{a^2-2ab+2(a-b)\sqrt{-b^2}}$.
    Ans. $\pm(a-b)\pm b\sqrt{-1}$.

11. Find the 3d power of $a\sqrt{-1}$. *Ans.* $-a^3\sqrt{-1}$.

12. Find the 3d power of $a - b\sqrt{-1}$.

Ans. $a^3 + b^3\sqrt{-1} - 3ab\left(b + a\sqrt{-1}\right)$.

13. Find the 4th power of $a + \sqrt{-b}$.

Ans. $a^4 - 6a^2b + b^2 + (4a^3 - 4ab)\sqrt{-b}$.

14. Find the values of x and y in the equation

$$2 + y + x\sqrt{-5} = 5 + x + y\sqrt{-2}.$$

Ans. $\begin{cases} x = 2 + \sqrt{10}, \\ y = 5 + \sqrt{10}. \end{cases}$

RADICAL EQUATIONS.

347. ***A Radical Equation*** is one which involves one or more radical quantities.

348. To free a radical equation from radical quantities.

EXAMPLES.

1. Free the equation

$$\sqrt{x} - \sqrt{3} = 2 \quad . \quad . \quad . \quad (1)$$

from radical quantities.

Transposing $\sqrt{3}$ to the second member, and squaring the resulting equation,

$$x = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3} \quad . \quad . \quad . \quad (2).$$

Transposing 7 in (2) to the first member, and squaring the resulting equation,

$$x^2 - 14x + 49 = 48;$$

whence, $\qquad x^2 - 14x = -1 \quad . \quad . \quad . \quad (3).$

2. Free the equation

$$\sqrt{x + 11} + \sqrt{x - 4} = 5 \quad . \quad . \quad . \quad (1)$$

from radical quantities, and find the value of x.

Transposing $\sqrt{x-4}$ to the second member, and squaring the resulting equation,

$$x + 11 = 25 - 10\sqrt{x-4} + x - 4;$$

whence, $\sqrt{x-4} = 1 \quad . \quad . \quad . \quad (2).$

Squaring (2), $x - 4 = 1$; whence, $x = 5$.

3. Free the equation

$$\frac{\sqrt{x} - \sqrt{x-5}}{\sqrt{x} + \sqrt{x-5}} = \frac{4x - 35}{5} \quad . \quad . \qquad (1)$$

from radical quantities, and find the value of x.

$$\frac{\sqrt{x} - \sqrt{x-5}}{\sqrt{x} + \sqrt{x-5}} = \frac{2x - 5 - 2\sqrt{x^2 - 5x}}{5} \ \textbf{(323)};$$

hence (1) becomes

$$2x - 5 - 2\sqrt{x^2 - 5x} = 4x - 35;$$

whence, $\sqrt{x^2 - 5x} = 15 - x \quad . \quad . \quad . \quad (2).$

Squaring (2), $x^2 - 5x = 225 - 30x + x^2$;

whence, $x = 9$.

4. Free the equation

$$\frac{c}{\sqrt{x} + \sqrt{a}} + \frac{m\sqrt{a}}{x - a} = \frac{m}{\sqrt{x} - \sqrt{a}} \quad . \quad . \quad . \quad (1)$$

from radical quantities, and find the value of x.

Multiplying both members of (1) by $x - a$,

$$c(\sqrt{x} - \sqrt{a}) + m\sqrt{a} = m(\sqrt{x} + \sqrt{a});$$

whence, $(c - m)\sqrt{x} = c\sqrt{a} \quad . \quad . \quad . \quad (2).$

Squaring (2), $(c - m)^2 x = ac^2$;

whence, $$x = \frac{ac^2}{(c - m)^2}.$$

Find the value of x in each of the following equations:

5. $\sqrt{x+7}+\sqrt{x}=7.$ *Ans.* $x=9.$

6. $x+3=\sqrt{x^2-4x+59}.$ *Ans.* $x=5.$

7. $\sqrt{\sqrt{x+48}-\sqrt{x}}=\sqrt[4]{x}.$ *Ans.* $x=16.$

8. $\sqrt[6]{x+2\sqrt{a+x}}=\sqrt[3]{a-\sqrt{a+x}}.$ *Ans.* $x=\frac{a^2-4a}{4}.$

9. $\frac{a}{\sqrt{x}}+\frac{\sqrt{x}}{c}=\sqrt{\frac{n}{x}}.$ *Ans.* $x=c(\sqrt{n}-a).$

10. $\frac{\sqrt{1-x}}{\sqrt{1+x}}+\frac{1+x}{\sqrt{1-x^2}}=\frac{3x}{\sqrt{1-x^2}}.$ *Ans.* $x=\frac{2}{3}.$

11. $\sqrt{c+x}=\frac{\sqrt{a+x^2}}{\sqrt{c+x}}.$ *Ans.* $x=\frac{a-c^2}{2c}.$

12. $x+\sqrt{c^2-ax}=\frac{c^2}{\sqrt{c^2-ax}}.$ *Ans.* $x=\frac{c^2-a^2}{a}.$

13. $\frac{1}{x}+\frac{1}{5}=\sqrt{\frac{1}{25}+\frac{1}{x}\sqrt{\frac{1}{5}+\frac{1}{x^2}}}.$ *Ans.* $x=20.$

14. $\sqrt{a-x}=\sqrt[4]{a+x^2}.$ *Ans.* $x=\frac{a-1}{2}.$

15. $\frac{\sqrt[4]{1+x}}{\sqrt{2-\sqrt{x}}}=\frac{\sqrt{2+\sqrt{x}}}{\sqrt[4]{4+x}}.$ *Ans.* $x=\frac{12}{13}.$

16. $\sqrt[3]{5+x}+\sqrt[3]{5-x}=\sqrt[3]{10}.$ *Ans.* $x=5.$

17. $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{a+x}}.$ *Ans.* $x=\frac{a}{3}.$

18. $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}.$ *Ans.* $x=\frac{b^2-4a^2}{4a}.$

19. $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$. *Ans.* $x=6$.

20. $\sqrt[3]{64+x^2-8x}=\dfrac{4+x}{\sqrt[3]{4+x}}$. *Ans.* $x=3$.

21. $\sqrt{5+x}+\sqrt{x}=\dfrac{15}{\sqrt{5+x}}$. *Ans.* $x=4$.

22. $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\dfrac{3}{2}\left(\dfrac{x}{x+\sqrt{x}}\right)^{\frac{1}{2}}$. *Ans.* $x=\dfrac{25}{16}$.

23. $\dfrac{\sqrt{ax}-b}{\sqrt{ax}+b}=\dfrac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$. *Ans.* $x=\dfrac{9b^2}{a}$.

24. $\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9$. *Ans.* $x=\dfrac{4}{9}$.

25. $\dfrac{3\sqrt{x}-4}{\sqrt{x}+2}=\dfrac{3\sqrt{x}+15}{\sqrt{x}+40}$. *Ans.* $x=4$.

349. SYNOPSIS FOR REVIEW.

CHAPTER XII. RADICAL QUANTITIES.

- DEFINITIONS . .
  - *Simple radical quantity.*
  - *Radical factor and its coefficient.*
  - *Degree of simple radical quantity.*
  - *Similar radical quantities.*
  - *Simplest form of radical quantity.*
  - *Rational quantity.*
  - *Irrational quantity.*
- REDUCTION . . .
  - *To reduce rational quantity to radical quantity of n^{th} degree.* Rule.
  - *To introduce coefficient of radical factor under radical sign.* Rule.
  - *To remove a factor from under the radical sign to the coefficient.* Rule.
  - *To reduce the indicated root of a fraction to an equivalent expression in which the quantity under the radical sign shall be entire.* Rules. Cor.

SYNOPSIS FOR REVIEW—*Continued.*

CHAPTER XII.—RADICAL QUANTITIES—*Continued.*

- REDUCTION—*Cont'd.*
  - *To reduce simple radical quantity to simplest form.* Rules.
  - *To reduce radical quantity of the form of* $\sqrt[mn]{a^n}$ *to another of lower degree.* Rule.
  - *To reduce simple radical quantity to another of higher or lower degree.* Rule. Cor. 1, 2.
  - *To reduce simple radical quantities having unequal indices to equivalent ones having equal indices.* Rule.
- COMBINATIONS . . . .
  - *To find the sum of simple radical quantities.* Rules.
  - *To find the difference of two simple radical quantities.* Rules.
  - *To find the product of two or more simple radical quantities.* Rules. Cor. 1, 2.
  - *To find the product of polynomial radical quantities.*
  - *To find the quotient of two simple radical quantities.* Rules. Cor. 1, 2.
  - *To find the quotient of polynomial radical quantities.*
- INVOLUTION OF RADICAL QUANTITIES.
  - *To raise the indicated* n^{th} *root of a quantity to any power.* Rule. Cor. 1, 2, 3.
  - *To raise a simple radical quantity to any power.* Rule. Cor.
  - *To raise a polynomial radical quantity to any power.*
- EVOLUTION OF RADICAL QUANTITIES.
  - *To find any root of the indicated root of a quantity.* Rules. Cor.
  - *To find any root of a simple radical quantity.* Rules. Cor.
  - *To find the square root or cube root of a polynomial radical quantity.*
- REDUCTION OF FRACTIONS HAVING SURD DENOMINATORS TO EQUIVALENT ONES HAVING RATIONAL DENOMINATORS.
  - *A simple surd.*
  - *A polynomial surd.*
  - *To reduce a fraction whose denominator is a simple surd to an equivalent one having a rational denominator.* Rule.
  - *To reduce a fraction whose denominator is a binomial surd to an equivalent one having a rational denominator.* Rules. Cor. 1, 2.
  - *Utility of preceding transformations.*

SYNOPSIS FOR REVIEW—*Continued.*

CHAPTER XII.—RADICAL QUANTITIES—*Continued.*

- **Prop. relating to Irrational Quantities.**
  - **325.**
  - **326.**
  - **327.**
  - **328.**
  - **329.**
  - **330.**
  - **331.**
  - **332.**
- **Simplifica'n of Complex Radical Quan.**
  - *A complex radical quantity.*
  - *To simplify* . . . .
    - $\sqrt{a \pm \sqrt{b}}.$
    - $\sqrt{a\sqrt{c} \pm \sqrt{b}}.$
    - $\sqrt[n]{a\sqrt[m]{b}}.$
    - $\sqrt{a\sqrt[n]{b} \pm c\sqrt[m]{d} \pm \text{etc.}}$
    - $\sqrt[3]{a\sqrt[n]{b} \pm c\sqrt[m]{d} \pm \text{etc.}}$
- **Imaginary Quantities.**
  - *An imaginary quantity.*
  - *A real quantity.*
  - *Classification of imaginary quantities.*
  - *Combinations of imaginary quantities.*
    - To find the sum.
    - To find the difference.
    - To find the product.
    - To find the quotient.
  - *To find all the powers of* $\sqrt{-1}.$

CHAPTER XIII.

QUADRATIC EQUATIONS WITH ONE UNKNOWN QUANTITY.—QUADRATIC EXPRESSIONS.

DEFINITIONS AND PRINCIPLES.

350. An equation which contains only one unknown quantity as x, and whose members are *entire* and *rational* with reference to x, is of the ***Second Degree*** when it contains x^2 and does not contain a higher power of x. Thus,

$$7x^2 = 3x + 160$$

is an equation of the *second* degree.

351. ***A Quadratic Equation*** is an equation of the second degree.

352. An equation of the second degree containing only one unknown quantity as x, when expressed in such a form that its members are *entire* and *rational* with reference to x, cannot have more than three kinds of terms, namely: terms which contain the square of x, terms which contain its first power, and known terms, that is, terms independent of x. Therefore, by transposing and uniting terms, the equation can be made to take the form of

$$ax^2 + bx = c;$$

a, b, and c being given quantities, which may be either positive or negative. For example, the equation

$$3x - \frac{2}{5} + \frac{x^2}{9} = 8 + \frac{2x^2}{3} - \frac{26x}{15},$$

can be transformed successively into the following equations:

$$\frac{x^2}{9} - \frac{2x^2}{3} + 3x + \frac{26x}{15} = 8 + \frac{2}{5},$$

$$5x^2 - 30x^2 + 135x + 78x = 360 + 18,$$

$$-25x^2 + 213x = 378,$$

$$25x^2 - 213x = -378.$$

We may consider a in the general equation as positive; for, if it is negative, we may make it positive by changing the signs of all the terms of the equation, as in the preceding example.

353. ***A Complete Equation of the Second Degree*** is one which can be expressed in the form of

$$ax^2 + bx = c,$$

in which neither b nor c is zero. Thus, $x^2 + 5x = 24$ and $2x^2 - 3x = 2x + 12$ are complete equations of the second degree.

The coefficient a cannot be zero; for then the equation would cease to be of the second degree.

A complete equation of the second degree is sometimes called an *Affected Quadratic Equation.*

354. If b or c is zero, the equation takes one of the forms

$$ax^2 = c, \quad ax^2 + bx = 0.$$

In either case the equation is said to be ***Incomplete.*** Thus, $3x^2 = 27$ and $2x^2 - 6x = 0$ are incomplete equations of the second degree.

An incomplete equation of the second degree of the form of $ax^2 = c$ is sometimes called a *Pure Quadratic Equation.*

INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

355. To solve an equation of the form of $ax^2 = c$.

Dividing both members of the equation by a, and extracting the square root of both members of the resulting equation, we find

$$x = \pm\sqrt{\frac{c}{a}}.$$

RULE.

Find the value of the square of the unknown quantity by the rule for solving a simple equation; the result will be an equation of the form of $x^2 = q$*; then extract the square root of both members of this equation.*

COR.—The two roots of a *pure quadratic equation* have equal *absolute* values, but contrary signs.

EXAMPLES.

Solve the following equations:

1. $\dfrac{3x^2+5}{8} - \dfrac{x^2+21}{3} = 39 - 5x^2.$ *Ans.* $x = \pm 3.$

2. $\dfrac{x^2}{2} = 14 - 3x^2.$ *Ans.* $x = \pm 2.$

3. $x^2 + 5 = \dfrac{10x^2}{3} - 16.$ *Ans.* $x = \pm 3.$

4. $(x+2)^2 = 4x + 5.$ *Ans.* $x = \pm 1.$

5. $\dfrac{3}{1+x} + \dfrac{3}{1-x} = 8.$ *Ans.* $x = \pm \dfrac{1}{2}.$

6. $\dfrac{3}{4x^2} - \dfrac{1}{6x^2} = \dfrac{7}{3}.$ *Ans.* $x = \pm \dfrac{1}{2}.$

7. $8x + \dfrac{7}{x} = \dfrac{65x}{7}.$ *Ans.* $x = \pm 2\frac{1}{3}.$

8. $\dfrac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x} = \dfrac{b}{c}.$ *Ans.* $x = \pm \dfrac{a(b-c)}{2\sqrt{bc}}.$

9. $x + \sqrt{a^2+x^2} = \dfrac{na^2}{\sqrt{a^2+x^2}}.$ *Ans.* $x = \pm \dfrac{a(n-1)}{\sqrt{2n-1}}.$

10. $\dfrac{\sqrt{a^2-x^2}-\sqrt{b^2+x^2}}{\sqrt{a^2-x^2}+\sqrt{b^2+x^2}} = \dfrac{c}{d}.$

Ans. $x = \pm \sqrt{\dfrac{a^2(c-d)^2 - b^2(c+d)^2}{2(c^2+d^2)}}.$

The following equations have such a form that they may be solved by a method similar to that employed in the solution of equations of the form of $ax^2 = c$:

11. $$\frac{28x}{x+18} = \frac{63(x+18)}{4x} \quad . \quad . \quad . \quad (1).$$

Clearing of fractions, (1) becomes

$$112x^2 = 63(x+18)^2 \quad . \quad . \quad . \quad (2).$$

Dividing both members of (2) by 7,

$$16x^2 = 9(x+18)^2 \quad . \quad . \quad . \quad (3).$$

Extracting the square root of both members of (3),

$$4x = \pm 3(x+18) \quad . \quad . \quad . \quad (4);$$

whence, $x = 54$ or $-7\frac{5}{7}$.

12. $(x-a)^2 = b$. *Ans.* $x = a \pm \sqrt{b}$.

356. To solve an equation of the form of $ax^2 + bx = 0$.

This equation may be expressed thus:

$$x(ax+b) = 0 \quad . \quad . \quad . \quad (1).$$

Now, in order that the product of x and $ax + b$ may be equal to zero, we must have

either $$x = 0 \quad . \quad . \quad . \quad (2),$$

or $$ax + b = 0 \quad . \quad . \quad . \quad (3).$$

From (3), $$x = -\frac{b}{a}.$$

Hence, an equation of the form of $ax^2 + bx = 0$ has roots, one of which is zero.

EXAMPLES.

Solve the following equations:

1. $2x^2 - \frac{19x}{2} = 8\frac{1}{2}x$. *Ans.* $x = 0$ or 9.

2. $x^2 - \frac{3}{2}x = \frac{5}{2}x.$ *Ans.* $x = 0$ or $4.$

3. $2 + \frac{x}{x^2} - \frac{2}{x} = 0.$

4. $x(2x + 5) = x(3x - 9).$

357. ***PROBLEMS.***

1. Find two numbers, one of which is four times the other, and the sum of whose squares is 153. *Ans.* ± 3 and ± 12.

2. Find two numbers, one of which is three times the other, and the difference of whose squares is 32. *Ans.* ± 2 and ± 6.

3. Find two numbers, one of which is three times as great as the other, and whose product is 75. *Ans.* ± 5 and ± 15.

4. A merchant bought two pieces of cloth, which together measured 36 yards. Each piece cost as many dimes a yard as there were yards in the piece, and the entire cost of one piece was four times that of the other. How many yards were there in each piece?

Ans. 24 yds. in one, and 12 yds. in the other.

The negative numbers are not given because they do not satisfy the question in its arithmetical sense.

5. Two persons, A and B, set out from different places to meet each other. They started at the same time, and traveled on the direct road between the two places. On meeting, it appeared that A had traveled 18 miles more than B; and that A could have traveled B's distance in $15\frac{3}{4}$ days, but that B would have been 28 days in traveling A's distance. Find the distance between the two places. *Ans.* 126 miles.

6. The product of the sum and difference of two numbers is 8, and the product of the sum of their squares and the difference of their squares is 80. What are the numbers?

Ans. ± 1 and ± 3.

7. The product of the sum and difference of two numbers is a, and the product of the sum of their squares and the difference of their squares is ma. What are the numbers?

$$Ans.\ \pm\sqrt{\frac{m-a}{2}} \quad \text{and} \quad \pm\sqrt{\frac{m+a}{2}}.$$

8. Two workmen, A and B, were engaged to work for a certain number of days at different wages. At the end of the time, A, who had been idle a of those days, received m dollars, and B, who had been idle b of those days, received n dollars. Now, if B had been idle a days, and A had been idle b days, they would have received equal amounts. For how many days were they engaged?

$$Ans.\ \frac{b\sqrt{m}-a\sqrt{n}}{\sqrt{m}-\sqrt{n}} \text{ days.}$$

COMPLETE EQUATIONS OF THE SECOND DEGREE.

358. To solve a complete equation of the second degree.

Let us consider the complete equation

$$ax^2+bx=c \quad . \quad . \quad . \quad (1).$$

Dividing both members of (1) by a,

$$x^2+\frac{b}{a}x=\frac{c}{a} \quad . \quad . \quad . \quad (2).$$

Let $p=\frac{b}{a}$, and $q=\frac{c}{a}$; then (2) becomes

$$x^2+px=q \quad . \quad . \quad . \quad (3).$$

Adding $\frac{p^2}{4}$ to both members of (3),

$$x^2+px+\frac{p^2}{4}=q+\frac{p^2}{4} \quad . \quad . \quad . \quad (4).$$

The first member of (4) is the square of $\left(x+\frac{p}{2}\right)$; hence, extracting the square root of both members,

$$x+\frac{p}{2}=\pm\sqrt{q+\frac{p^2}{4}} \quad . \quad . \quad . \quad (5).$$

Transposing $\frac{p}{2}$,

$$x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}} \quad . \quad . \quad . \quad (6).$$

The given equation, therefore, has two roots, namely:

$$-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}},$$

$$-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}.$$

The operation of transforming (3) into (4) is called *Completing the Square.*

RULE.

I. *Reduce the given equation to the form of* $x^2+px=q$.

II. *Add to both members of this equation the square of half the coefficient of* x.

III. *Extract the square root of both members of the equation thus obtained; the result will be an equation of the form of* $x+\frac{p}{2}=\pm m$, *from which the values of* x *may be found by transposition.*

EXAMPLES.

Solve the following equations:

1 $$-3x^2+36x=105.$$

Dividing both members by -3,

$$x^2-12x=-35.$$

Completing the square,

$$x^2-12x+36=-35+36=1.$$

Extracting the square root of both members,

$$x - 6 = \pm 1;$$

$\therefore$ $$x = 6 \pm 1;$$

that is, $$x = 7 \text{ or } 5.$$

Verification. $-3 \times 7^2 + 36 \times 7 = -147 + 252 = 105;$
$-3 \times 5^2 + 36 \times 5 = -\ \ 75 + 180 = 105.$

2. $x^2 - 4x + 3 = 0.$ *Ans.* $x = 1$ or $3.$

3. $6x^2 - 13x = -6.$ *Ans.* $x = \frac{2}{3}$ or $\frac{3}{2}.$

4. $x^2 - 5x + 4 = 0.$ *Ans.* $x = 1$ or $4.$

5. $3x^2 - 7x = 20.$ *Ans.* $x = 4$ or $-\frac{5}{3}.$

6. $2x^2 - 7x + 3 = 0.$ *Ans.* $x = 3$ or $\frac{1}{2}.$

7. $3x^2 - 53x + 34 = 0.$ *Ans.* $x = 17$ or $\frac{2}{3}.$

8. $x^2 + 10x + 24 = 0.$ *Ans.* $x = -4$ or $-6.$

9. $(x-1)(x-2) = 6.$ *Ans.* $x = 4$ or $-1.$

10. $(3x-5)(2x-5) = (x+3)(x-1).$
Ans. $x = 4$ or $\frac{7}{5}.$

11. $(2x-3)^2 = 8x.$ *Ans.* $x = \frac{9}{2}$ or $\frac{1}{2}.$

12. $\frac{48}{x+3} = \frac{165}{x+10} - 5.$ *Ans.* $x = 5\frac{2}{5}$ or $5.$

13. $\sqrt{(2x+7)} + \sqrt{(3x-18)} = \sqrt{(7x+1)}.$
Ans. $x = 9$ or $-3\frac{3}{5}.$

14. $ax^2 - ac = cx - bx^2.$ *Ans.* $x = \frac{c \pm \sqrt{4a^2c + 4abc + c^2}}{2(a+b)}.$

15. $a^2 + b^2 - 2bx + x^2 = \frac{m^2x^2}{n^2}.$
Ans. $x = \frac{n}{n^2 - m^2}\left(bn \pm \sqrt{a^2m^2 + b^2m^2 - a^2n^2}\right).$

16. $3x^{-2} + 2x^{-1} = 1.$ *Ans.* $x = 3$ or $-1.$

17. $\frac{5}{x^{-2}} - \frac{3}{x^{-1}} = 2x^0.$ *Ans.* $x = 1$ or $-\frac{2}{5}.$

18. $3x + 2\sqrt{x} = 16.$ *Ans.* $x = 7\frac{1}{9}$ or $4.$

19. $\sqrt{x+5} = \frac{12}{\sqrt{x+12}}.$ *Ans.* $x = 4$ or $-21.$

20. $\sqrt{x} + \sqrt{a-x} = \sqrt{b}.$ *Ans.* $x = \frac{a \pm \sqrt{2ab - b^2}}{2}.$

21. $\sqrt{x+m} - \sqrt{x+n} = \sqrt{2x}.$

Ans. $x = -\frac{m+n}{2} \pm \frac{1}{2}\sqrt{2m^2 + 2n^2}.$

22. $(x-c)(ab)^{\frac{1}{2}} - \frac{(a-b)}{(cx)^{-\frac{1}{2}}} = 0.$ *Ans.* $x = \frac{ac}{b}$ or $\frac{bc}{a}.$

23. $\frac{1}{(a+x)^{-\frac{1}{2}}} + \frac{1}{(a-x)^{-\frac{1}{2}}} = \frac{12a(a+x)^{-\frac{1}{2}}}{5}.$

Ans. $x = \frac{4a}{5}$ or $\frac{3a}{5}.$

24. $\frac{x - \sqrt{x+1}}{x + \sqrt{x+1}} = \frac{5}{11}.$ *Ans.* $x = 8$ or $-\frac{8}{9}.$

25. $\frac{1}{ax^{-1}} + bx^{-1} = \frac{1}{c^{-1}}.$ *Ans.* $x = \frac{ac \pm \sqrt{a^2c^2 - 4ab}}{2}.$

359. When a complete equation of the second degree is proposed for solution, instead of going through the process of completing the square, we may use the *formula* $x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$ For example, take the equation

$$-3x^2 + 36x = 105.$$

Dividing by -3, $\quad x^2 - 12x = -35.$

In this case, $p = -12$, and $q = -35$; hence, by the formula,

$$x = -\frac{-12}{2} \pm \sqrt{-35 + \frac{(-12)^2}{4}} = 7 \text{ or } 5.$$

EXAMPLES.

Solve the following equations by using the formula

$$x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}:$$

1. $x^2 - 6x = 7.$ *Ans.* $x = 7$ or -1.

2. $x^2 + 14x = 95.$ *Ans.* $x = 5$ or -19.

3. $x^2 - 2x = 8.$ *Ans.* $x = 4$ or -2.

4. $x^2 + 10x = -9.$ *Ans.* $x = -1$ or -9.

5. $x^2 - 14x = 120.$ *Ans.* $x = 20$ or -6.

6. $x^2 + 32x = 320.$ *Ans.* $x = 8$ or -40.

7. $x^2 + 100x = 1100.$ *Ans.* $x = 10$ or -110.

8. $x^2 - x = \frac{3}{4}.$ *Ans.* $x = 1\frac{1}{2}$ or $-\frac{1}{2}$.

9. $x^2 + 3\frac{1}{3}x = 19.$ *Ans.* $x = 3$ or $-6\frac{1}{3}$.

10. $x^2 + \frac{13}{5}x = 74.$ *Ans.* $x = 7\frac{2}{5}$ or -10.

11. $2x = 4 + \frac{6}{x}.$ *Ans.* $x = 3$ or -1.

12. $x - \frac{x^3 - 8}{x^2 + 5} = 2.$ *Ans.* $x = 2$ or $\frac{1}{2}$.

13. $\frac{x^2}{3m - 2a} - \frac{x}{2} = \frac{m^2 - 4a^2}{4a - 6m}.$

Ans. $x = m - 2a$ or $\frac{1}{2}m + a$.

14. $\frac{x+3}{x-3} - \frac{x-3}{x+3} = a.$ *Ans.* $x = \frac{3}{a}(2 \pm \sqrt{a^2 + 4})$.

15. $mx^2 - \frac{m^3 - n^2}{mn}x = 1.$ *Ans.* $x = \frac{m}{n}$ or $-\frac{n}{m^2}$.

16. $\frac{8}{9}[x^2 + a(a + b)] + \frac{1}{3}bx = \frac{1}{9}x(20a + 7b).$

Ans. $x = 2a$ or $\frac{1}{2}(a + b)$.

360. Solving the equation $ax^2 + bx = c$ in the usual way, we find

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

To solve an equation of the second degree by means of this formula, it is only necessary to reduce it to the form of $ax^2+bx=c$, and then make the proper substitutions. For example, take the equation $-3x^2+36x=105$. In this example, $a=-3$, $b=36$, and $c=105$; hence, by the formula,

$$x = \frac{-36 \pm \sqrt{36^2 + 4(-3)105}}{-6} = \frac{-36 \pm \sqrt{1296 - 1260}}{-6} =$$

$$\frac{-36 \pm \sqrt{36}}{-6} = \frac{-36 \pm 6}{-6} = 6 \mp 1 = 5 \text{ or } 7.$$

Let the student solve some of the equations of Articles **358** and **359** in this way.

361. To complete the square by the Hindoo Method.

Take the equation

$$ax^2 + bx = c \quad . \quad . \quad . \quad (1).$$

Multiplying both members by $4a$,

$$4a^2x^2 + 4abx = 4ac \quad . \quad . \quad . \quad (2).$$

Adding b^2 to both members of (2),

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2 \quad . \quad . \quad . \quad (3).$$

Extracting the square root of both members of (3),

$$2ax + b = \pm\sqrt{4ac + b^2} \quad . \quad . \quad . \quad (4);$$

whence, $$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a} \quad . \quad . \quad . \quad (5).$$

Solve by this method the following equation:

$$5x^2 - 3x = 224.$$

Multiplying both members by 4×5, the given equation becomes

$$100x^2 - 60x = 4480.$$

Adding 3^2 to both members of this equation,

$$100x^2 - 60x + 9 = 4489.$$

Extracting the square root,

$$10x - 3 = \pm 67;$$

whence, $$x = \frac{3 \pm 67}{10} = 7 \text{ or } -6\tfrac{2}{5}.$$

Let the student solve some other equations in this way.

362. To cause the term containing the first power of the unknown quantity to disappear.

If, in the equation

$$x^2 + px = q,$$

we substitute $z - \frac{p}{2}$ for x, we obtain

$$\left(z - \frac{p}{2}\right)^2 + p\left(z - \frac{p}{2}\right) = q;$$

that is, $$z^2 - \frac{p^2}{4} = q;$$

whence, $$z = \pm\sqrt{q + \frac{p^2}{4}};$$

$\therefore$ $$x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

Solve by this method the following equation:

$$x^2 - 11x = -18.$$

Substituting $z + \frac{11}{2}$ for x, this equation becomes

$$\left(z + \frac{11}{2}\right)^2 - 11\left(z + \frac{11}{2}\right) = -18;$$

whence, $$z^2 = \frac{49}{4},$$

and $$z = \pm\frac{7}{2};$$

$\therefore$ $$x = \frac{11}{2} \pm \frac{7}{2} = 9 \text{ or } 2.$$

363. *PROBLEMS.*

1. By selling my horse for 24 dollars, I lose as much per cent as the horse cost me. What did I pay for the horse?

Let $x =$ the number of dollars that I paid for the horse; then will $\frac{x}{100} \times x =$ my loss. But $x - 24$ also expresses this loss.

$\therefore$ $$\frac{x^2}{100} = x - 24;$$

whence, $$x = 60 \text{ or } 40.$$

2. Divide the number 10 into two parts, such that their product shall be 24.

Let $$x = \text{one part};$$

then will $$10 - x = \text{the other part.}$$

Hence, by the problem,

$$x(10 - x) = 24;$$

whence, $$x = 4 \text{ or } 6;$$

therefore $$10 - x = 6 \text{ or } 4.$$

Here, although x may have either of two values, yet there is only one answer to the problem; one part must be 4 and the other 6.

3. A person bought a certain number of oxen for \$400. If he had bought 4 more for the same sum, each ox would have cost \$5 less. How many did he buy?

Let $\qquad x =$ the uumber of oxen;

then will $\qquad \frac{400}{x} =$ the cost of each in dollars.

If he had bought 4 more for the same sum, the cost of each would have been $\frac{400}{x+4}$;

$$\therefore \qquad \frac{400}{x+4} = \frac{400}{x} - 5;$$

whence, $\qquad x = 16$ or -20.

Only the positive value of x is admissible; hence, the number of oxen is 16.

In solving problems by algebra, results will sometimes be obtained which do not apply to the question actually proposed. The reason is that the algebraic language is more general than ordinary language, and thus the equation, which is a proper expression of the *conditions* of the problem, is also applicable to other conditions. It is sometimes possible, by making suitable changes in the enunciation of the original problem, to form a new problem, corresponding to any result which was inapplicable to the original problem. If we change the sign of x in the equation $\frac{400}{x+4} = \frac{400}{x} - 5$, it becomes $\frac{400}{4-x} = \frac{400}{-x} - 5$, or $\frac{400}{x-4} = \frac{400}{x} + 5$.

This equation is the algebraic statement of the following problem: A person bought a certain number of oxen for \$400. If he had bought 4 less for the same sum, each ox would have cost \$5 more. How many did he buy?

Solving the equation $\frac{400}{x-4} = \frac{400}{x} + 5$, we find $x = 20$ or -16.

In this connection the student should review Art. **216.**

4. Find two numbers whose difference is 8 and whose product is 240. *Ans.* 12 and 20.

5. Find two numbers whose difference is $2a$ and whose product is b. *Ans.* $a \pm \sqrt{a^2 + b}$ and $-a \pm \sqrt{a^2 + b}$.

6. The remainder obtained by subtracting a certain number from 10 is equal to the quotient obtained by dividing 25 by that number. What is the number? *Ans.* 5.

7. Divide the number 40 into two such parts that their product shall be equal to 15 times their difference.

Ans. 60 and -20, or 10 and 30.

The numbers 60 and -20 satisfy the problem in the *algebraic* sense, but not in the *arithmetical* sense.

8. Divide a into two such parts that their product shall be equal to m times their difference.

Ans. $\dfrac{a-2m \pm \sqrt{a^2+4m^2}}{2}$ and $\dfrac{a+2m \mp \sqrt{a^2+4m^2}}{2}$.

9. Divide 100 into two such parts that the sum of their square roots shall be 14. *Ans.* 64 and 36.

10. Divide a into two such parts that the sum of their square roots shall be s.

Ans. $\dfrac{a+\sqrt{2as^2-s^4}}{2}$ and $\dfrac{a-\sqrt{2as^2-s^4}}{2}$.

11. A and B start at the same time from different places and travel toward each other. At the end of 14 hours they meet, when it appears that A has traveled 10 miles more than B, and that their rates of travel are such that B requires half an hour more than A to travel 20 miles. Find B's rate of travel.

Ans. 5 miles.

The negative result is rejected, because it does not satisfy the problem in its arithmetical sense.

12. A and B start at the same time from different places and travel toward each other. At the end of m hours they meet, when it appears that A has traveled a miles more than B, and that their rates of travel are such that B requires n hours more than A to travel b miles. Find B's rate of travel.

Ans. $-\dfrac{a}{2m}+\sqrt{\dfrac{ab}{mn}+\dfrac{a^2}{4m^2}}$.

13. A started from C toward D, and traveled at the rate of 10 miles an hour. When he was 9 miles from C, B started from D toward C, and went every hour one-twentieth of the distance from D to C. When B had traveled as many hours as he went miles in one hour, he met A. Find the distance from C to D.

Ans. 180 miles or 20 miles.

14. A went from C to D, traveling a miles an hour. When he was b miles from C, B started from D toward C, and went every hour $\frac{1}{n}$th of the distance from D to C. When B had traveled as many hours as he went miles in one hour, he met A. Find the distance from C to D.

$$\textit{Ans. } n\left[\frac{n-a}{2} \pm \sqrt{\left(\frac{n-a}{2}\right)^2 - b}\right].$$

15. A and B were traveling on the same road, and at the same rate, from Columbia to St. Louis. At the 50th mile-stone from St. Louis, A overtook a flock of geese which were traveling at the rate of three miles in two hours, and two hours afterward met a wagon which was moving at the rate of nine miles in four hours. B overtook the same flock of geese at the 45th mile-stone, and met the same wagon 40 minutes before he reached the 31st mile-stone. Where was B when A reached St. Louis?

Ans. 25 miles from St. Louis.

THEORY OF QUADRATIC EQUATIONS WITH ONE UNKNOWN QUANTITY.

364. *Every equation of the second degree containing only one unknown quantity has two roots, and only two.*

Every equation of the second degree containing only one unknown quantity can be reduced to the form of

$$x^2 + px = q.$$

Solving this equation, we find

$$x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

Hence x has two values, namely: $-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}$ and $-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}$.

Denoting the first of these values by x' and the second by x'', we have

$$x'=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}},$$

$$x''=-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}.$$

The equation $x^2+px=q$ cannot have more than *two* roots.

If possible, let a, b, and c be three different roots of this equation; then will these roots satisfy the equation.

$$\therefore \quad a^2+pa=q \quad . \quad . \quad . \quad (1),$$

$$b^2+pb=q \quad . \quad . \quad . \quad (2),$$

$$c^2+pc=q \quad . \quad . \quad . \quad (3).$$

Subtracting (2) from (1),

$$a^2-b^2+p(a-b)=0 \quad . \quad . \quad . \quad (4).$$

Subtracting (3) from (1),

$$a^2-c^2+p(a-c)=0 \quad . \quad . \quad . \quad (5).$$

Dividing both members of (4) by $a-b$, which is, by hypothesis, not zero, we obtain

$$a+b+p=0 \quad . \quad . \quad . \quad (6).$$

Dividing both members of (5) by $a-c$,

$$a+c+p=0 \quad . \quad . \quad . \quad (7).$$

Subtracting (7) from (6),

$$b-c=0;$$

whence, $$b=c;$$

that is, two of the supposed roots are equal to each other; therefore the equation $x^2+px=q$ cannot have three different roots.

365. *The sum of the roots of an equation of the form of* $x^2+px=q$ *is equal to the coefficient of the second term taken with the contrary sign.*

Solving the equation $x^2+px=q$, we obtain

$$x' = -\frac{p}{2}+\sqrt{q+\frac{p^2}{4}} \quad . \quad . \quad . \quad (1),$$

and
$$x'' = -\frac{p}{2}-\sqrt{q+\frac{p^2}{4}} \quad . \quad . \quad . \quad (2).$$

whence, by addition,

$$x'+x''=-p.$$

Thus, the roots of the equation $x^2-10x=-16$ are 8 and 2, and their sum is 10.

366. *The product of the roots of an equation of the form of* $x^2+px=q$ *is equal to the second member taken with the contrary sign.*

From (1) and (2) of Art. **365** we obtain, by multiplication,

$$x'x'' = \left(-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}\right)\left(-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}\right)=\frac{p^2}{4}-\left(q+\frac{p^2}{4}\right)$$
$$=-q.$$

Thus, the roots of the equation $x^2-10x=-16$ are 8 and 2, and their product is 16.

Cor.—The independent term q is divisible by each of the roots.

367. *Every equation of the second degree containing only one unknown quantity can be reduced to the form of*

$$(x-x')(x-x'')=0.$$

Denoting the roots of the equation

$$x^2+px=q \quad . \quad . \quad . \quad (1),$$

by x' and x'', we have

$$p=-(x'+x''),$$

and
$$q=-x'x'';$$

hence (1) becomes

$$x^2 - (x' + x'')\,x = -x'x'' \quad . \quad . \quad . \quad (2).$$

By transposition and factoring, (2) becomes

$$(x - x')\,(x - x'') = 0 \quad . \quad . \quad . \quad (3).$$

COR.—Hence $x^2 + px - q = (x - x')\,(x - x'')$; therefore the first member of the equation $x^2 + px - q = 0$ is divisible by $x - x'$ and by $x - x''$.

368. *EXAMPLES.*

1. Find the equation whose roots are 2 and 3.

1st Solution.—Substituting $-(2+3)$ for p (**365**), and -2×3 for q (**366**), the general equation $x^2 + px = q$ becomes

$$x^2 - 5x = -6.$$

2d Solution.—Substituting 2 for x' and 3 for x'', (3) of Art. **367** becomes

$$(x - 2)\,(x - 3) = 0;$$

that is,
$$x^2 - 5x + 6 = 0.$$

2. Resolve the first member of the equation $x^2 + 6x + 8 = 0$ into two binomial factors.

Solving this equation, we find $x' = -2$, $x'' = -4$; hence the given equation may be written in the form

$$[x-(-2)]\,[x-(-4)] = 0 \ (\mathbf{367}),$$

that is,
$$(x + 2)\,(x + 4) = 0.$$

3. Find the equation whose roots are 5 and 2.

Ans. $x^2 - 7x = -10.$

4. Find the equation whose roots are 3 and 3.

Ans. $x^2 - 6x = -9.$

5. Find the equation whose roots are 10 and $-\frac{44}{3}$.

Ans. $x^2 + \frac{14}{3}x = \frac{440}{3}.$

6. Find the equation whose roots are $\frac{13+\sqrt{85}}{14}$ and $\frac{13-\sqrt{85}}{14}$.

Ans. $x^2-\frac{13}{7}x=-\frac{3}{7}$

7. Find the equation whose roots are $5+\sqrt{-1}$ and $5-\sqrt{-1}$.

Ans. $x^2-10x=-26$.

8. Resolve the first member of the equation $3x^2-10x-25=0$ into three factors.

Ans. $3(x-5)\left(x+\frac{5}{3}\right)=0$.

9. Resolve the first member of the equation $x^2+73x+780=0$ into two binomial factors.

Ans. $(x+60)(x+13)=0$.

10. Resolve the first member of the equation $2x^2+x-6=0$ into three factors.

Ans. $2(x+2)\left(x-\frac{3}{2}\right)=0$.

11. Resolve the first member of the equation $x^2-88x+1612=0$ into two binomial factors.

Ans. $(x-62)(x-26)=0$.

12. Resolve the first member of the equation $x^2+a^2=0$ into two binomial factors.

Ans. $\left(x-a\sqrt{-1}\right)\left(x+a\sqrt{-1}\right)=0$.

DISCUSSION OF THE EQUATION $x^2+px=q$.

369. ***The Discussion*** of an equation consists in making every possible supposition with regard to the arbitrary quantities contained in it, and interpreting the results.

The *arbitrary* quantities in the equation $x^2+px=q$ are p and q.

370. We shall first make every possible supposition in relation to the signs of p and q.

Suppose, *first*, that p and q are positive; *second*, that p is negative and q positive; *third*, that p is positive and q negative; *fourth*, that p and q are negative. We shall thus have

The Four Forms.

$$x^2 + px = \quad q \quad . \quad . \quad . \quad (1),$$
$$x^2 - px = \quad q \quad . \quad . \quad . \quad (2),$$
$$x^2 + px = -q \quad . \quad . \quad . \quad (3),$$
$$x^2 - px = -q \quad . \quad . \quad . \quad (4).$$

371. *In the first form one root is positive, the other negative, and the negative root is numerically the greater.*

Since q is positive, the product of the roots is negative (**366**); hence one root is positive and the other negative. Again, since p is positive, the sum of the roots is negative (**365**); hence the negative root is numerically the greater.

Illustration.—The roots of the equation $x^2 + x = 6$ are 2 and -3.

372. *In the second form one root is positive, the other negative, and the positive root is numerically the greater.*

Since q is positive, the product of the roots is negative; hence one root is positive and the other negative. Again, since p is negative, the sum of the roots is positive; hence the positive root is numerically the greater.

Illustration.—The roots of the equation $x^2 - x = 210$ are 15 and -14.

373. *In the third form both roots are negative.*

Since q is negative, the product of the roots is positive; hence they have *like* signs; and since p is positive, the sum of the roots is negative; hence both roots are negative.

Illustration.—The roots of the equation $x^2 + 7x = -12$ are -4 and -3.

374. *In the fourth form both roots are positive.*

Since q and p are negative, the product and the sum of the roots are positive; hence both roots are positive.

Illustration.—The roots of the equation $x^2 - 7x = -12$ are 4 and 3.

375. For convenient reference, the four forms and their corresponding roots are here given.

$$x^2 + px = q \quad . \quad . \quad . \quad (1); \text{ whence } \begin{cases} x' = -\frac{p}{2} + \sqrt{q + \frac{p^2}{4}}, \\ x'' = -\frac{p}{2} - \sqrt{q + \frac{p^2}{4}}. \end{cases}$$

$$x^2 - px = q \quad . \quad . \quad . \quad (2); \text{ whence } \begin{cases} x' = \frac{p}{2} + \sqrt{q + \frac{p^2}{4}}, \\ x'' = \frac{p}{2} - \sqrt{q + \frac{p^2}{4}}. \end{cases}$$

$$x^2 + px = -q \quad . \quad . \quad . \quad (3); \text{ whence } \begin{cases} x' = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}, \\ x'' = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}. \end{cases}$$

$$x^2 - px = -q \quad . \quad . \quad . \quad (4); \text{ whence } \begin{cases} x' = \frac{p}{2} + \sqrt{\frac{p^2}{4} - q}, \\ x'' = \frac{p}{2} - \sqrt{\frac{p^2}{4} - q}. \end{cases}$$

376. *Unequal Roots.*—The roots of an equation of the first or of the second form are unequal, whatever the relative values of p and q may be **(371–372)**.

The roots of an equation of the third or of the fourth form are unequal if $\frac{p^2}{4}$ is greater or less than q.

377. *Equal Roots.*—The roots of an equation of the third or of the fourth form are equal if $\frac{p^2}{4}$ is equal to q.

Illustration.—Solving the equation $x^2 + 6x = -9$, we find $x' = -3$ and $x'' = -3$. Solving the equation $x^2 - 6x = -9$, we find $x' = 3$ and $x'' = 3$.

378. *Real Roots.*—The roots of an equation of the first

or of the second form are *real*, whatever the relative values of p and q may be, for in these forms $\frac{p^2}{4}+q$ is positive.

The roots of an equation of the third or of the fourth form are *real* if $\frac{p^2}{4}$ is not less than q.

REMARK.—The quantities p and q are here supposed to be *real*.

379. *Imaginary Roots.*—The roots of an equation of the third or of the fourth form are *imaginary* if $\frac{p^2}{4}$ is less than q; for the radical part of each of the roots, in this case, is the square root of a negative quantity.

Illustration.—The roots of the equation $x^2+6x=-10$ are $-3+\sqrt{-1}$ and $-3-\sqrt{-1}$; and the roots of the equation $x^2-6x=-10$ are $3+\sqrt{-1}$ and $3-\sqrt{-1}$.

380. *Imaginary roots indicate incompatible conditions.*

The demonstration depends upon the following

LEMMA.—*The greatest product which can be obtained by separating a given number into two parts and multiplying one by the other is the square of half that number.*

Let p be the given number, and d the difference of the parts into which it is separated.

Then $\quad \frac{p}{2}+\frac{d}{2}=$ the greater part,

and $\quad \frac{p}{2}-\frac{d}{2}=$ the less part (**213,** 4).

Denoting the product of the parts by P, we have

$$P=\frac{p^2}{4}-\frac{d^2}{4}.$$

Now, since p is a given number, it is evident that P will in-

crease as d diminishes, and will be the greatest possible when $d = 0$; that is, P, when greatest, is equal to $\left(\frac{p}{2}\right)^2$.

Illustration.

$$8 = 1 + 7;\quad 7 \times 1 = 7.$$
$$8 = 2 + 6;\quad 6 \times 2 = 12.$$
$$8 = 3 + 5;\quad 5 \times 3 = 15.$$
$$8 = 4 + 4;\quad 4 \times 4 = 16.$$

In the first form the sum of the roots is $-p$ (**365**), and their product is $-q$ (**366**); hence (Lem.), in this form, $-q$ cannot be greater than $\frac{p^2}{4}$.

In the second form the sum of the roots is p, and their product is $-q$; hence, in this form, $-q$ cannot be greater than $\frac{p^2}{4}$.

In the third form the sum of the roots is $-p$, and their product is q; hence, in this form, q cannot be greater than $\frac{p^2}{4}$.

In the fourth form the sum of the roots is p, and their product is q; hence, in this form, q cannot be greater than $\frac{p^2}{4}$.

In the first and second forms q is *positive;* hence an equation in which $-q$ is greater than $\frac{p^2}{4}$ can never occur in either of these forms; for $\frac{p^2}{4}$ being positive is greater than any negative quantity. But in the third and fourth forms an equation *may* occur in which $+q$ is greater than $\frac{p^2}{4}$. Thus, in the equations $x^2 + 6x = -10$ and $x^2 - 6x = -10$, the independent term, taken with the contrary sign, is greater than the square of half the coefficient of the second term; therefore the roots are imaginary.

Hence, if any problem furnishes an equation of the third or of the fourth form, in which the independent term, taken with the contrary sign, is greater than the square of half the coefficient of

the second term, we infer that the problem contains *incompatible* conditions.

For example, let it be required to find two numbers whose sum shall be 6 and product 10.

Let $\qquad x =$ one of the numbers;

then will $\qquad 6 - x =$ the other.

By the second condition of the problem,

$$x(6 - x) = 10;$$

that is, $\qquad 6x - x^2 = 10,$

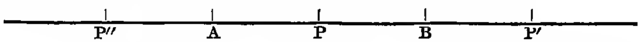
or, by changing signs, $\qquad x^2 - 6x = -10;$

whence, $\qquad x = 3 \pm \sqrt{-1}.$

The imaginary roots indicate that there are no *real* numbers whose sum is 6 and product 10. The greatest product which can be formed by separating 6 into two parts and multiplying one by the other is 9.

PROBLEM OF THE LIGHTS.

381. *To find, on the straight line joining two lights, the points which are equally illuminated by those lights.*



Let A and B be the two lights. Denote the intensity of the light A at a unit's distance by a, the intensity of the light B at a unit's distance by b, and the distance between the lights by d.

Let P be a point equally illuminated by the two lights, and let $x = \text{AP}$; then will $d - x = \text{BP}$.

One of the laws of light is, that the intensity of a light at any distance as x, is equal to the quotient obtained by dividing its intensity at a unit's distance by x^2; hence

$$\frac{a}{x^2} = \text{the intensity of the light A at P,}$$

and

$$\frac{b}{(d - x)^2} = \text{the intensity of the light B at P.}$$

But P is to be equally illuminated by the two lights;

$$\therefore \qquad \frac{a}{x^2} = \frac{b}{(d-x)^2} \quad . \quad . \quad . \quad (1).$$

Clearing this equation of fractions,

$$a(d-x)^2 = bx^2 \quad . \quad . \quad . \quad (2).$$

Extracting the square root of both members of (2),

$$(d-x)\sqrt{a} = \pm x\sqrt{b} \quad . \quad . \quad . \quad (3);$$

whence, $$x = d\left(\frac{\sqrt{a}}{\sqrt{a} \pm \sqrt{b}}\right).$$

Separating the values of x,

$$x' = d\left(\frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}}\right)$$

and $$x'' = d\left(\frac{\sqrt{a}}{\sqrt{a} - \sqrt{b}}\right).$$

From the nature of the problem, a and b are positive; hence the values of x are *real;* therefore there are two points of equal illumination, and only two, on the line of the lights.

Six different suppositions can be made upon the arbitrary quantities a, b, and d, namely:

1. $a > b$ and $d > 0$.	4. $a > b$ and $d = 0$.
2. $a = b$ and $d > 0$.	5. $a = b$ and $d = 0$.
3. $a < b$ and $d > 0$.	6. $a < b$ and $d = 0$.

1. $a > b$ *and* $d > 0$.

In this case $\dfrac{\sqrt{a}}{\sqrt{a} + \sqrt{b}}$ is a proper fraction; that is, it is less than 1; and since the denominator is less than twice the numerator, the fraction is greater than $\frac{1}{2}$.

$$\therefore \qquad d\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) < d \text{ and } > \frac{1}{2}d.$$

The point P is therefore between the two lights and nearer the weaker one.

The fraction
$$\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} > 1;$$

$$\therefore \qquad d\left(\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\right) > d.$$

The second point of equal illumination is, therefore, at some point P′ on the right of B.

2. $a = b$ *and* $d > 0$.

In this case $x' = \frac{d}{2}$ and $x'' = \frac{d\sqrt{a}}{0} = \infty$ (**222**, 1); that is, the first point of equal illumination is at the middle point of AB, and the second is at an infinite distance to the right of A. The symbol ∞ indicates *impossibility;* that is, it shows that there is no second point of equal illumination.

3. $a < b$ *and* $d > 0$.

In this case $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} < \frac{1}{2}$ and $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ is negative;

$$\therefore \qquad x' < \frac{1}{2}d \quad \text{and} \quad x'' < 0.$$

The first point of equal illumination is, therefore, between the lights and nearer to A, and the second point is at some point P″ on the left of A.

4. $a > b$ *and* $d = 0$.

In this case $x' = 0$ and $x'' = 0$.

How are these results to be interpreted? They *seem* to indi-

cate that the point at which the lights are placed is equally illuminated by them; but this is not true, as we shall see by considering equation (1). Under the hypothesis that $d = 0$, (1) becomes

$$\frac{a}{x^2} = \frac{b}{x^2}.$$

But this is not an equation in fact, for $a > b$, and the denominators are equal. It would not be an equation if $x = 0$, for then $\frac{a}{x^2}$ and $\frac{b}{x^2}$ become unequal infinities.

There is, then, in this case, no *equation*, and hence no point of equal illumination.

5. $a = b$ and $d = 0$.

In this case $x' = 0$ and $x'' = \frac{0}{0}$.

The first value of x indicates that the point at which the two equal lights are placed is equally illuminated by them, and the second value of x indicates that *any* point on the line of the lights is equally illuminated by them (**222**, 4). As the lights are now at the same point, the *line* of the lights may be drawn in any direction in space; hence, in this case, *any* point in space will be equally illuminated by the two lights.

6. $a < b$ *and* $d = 0$.

In this case $x' = 0$ and $x'' = 0$.

But under this hypothesis, as in case 4, equation (1) becomes *impossible;* hence no point in space is equally illuminated by the two lights.

QUADRATIC EXPRESSIONS.

382. ***A Quadratic Expression*** is a trinomial of the form of

$$ax^2 + bx + c;$$

in which a, b, and c represent given numbers, positive or negative, and in which x may have *any* value. Thus,

$$5x^2 - 3x + 6$$

is a quadratic expression.

A quadratic expression is sometimes called a *Trinomial of the Second Degree.*

383. *Distinction between a Quadratic Equation and a Quadratic Expression.* —In the quadratic *equation* $ax^2 + bx + c = 0$, x has one of two definite values (**364**); but in the quadratic *expression* $ax^2 + bx + c$, x may have *any* value.

384. To resolve the quadratic expression ax^2+bx+c into its factors.

Assume $$ax^2 + bx + c = 0;$$

whence, $$x' = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

and $$x'' = \frac{-b - \sqrt{b^2 - 4ac}}{2a};$$

$$\therefore \quad ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right).$$

RULE.

Assume the given quadratic expression to be equal to zero, and resolve the first member of the equation thus obtained into its factors (**367**).

EXAMPLES.

Resolve each of the following expressions into its prime factors:

1. $x^2 + 2x - 120$.
2. $x^2 - 9x + 14$.
3. $x^2 + 8x + 15$.
4. $8x^2 - 2x - 3$.
5. $6x^2 + x - 1$.
6. $x^2 + 9x - 90$.
7. $x^2 + 12x - 45$.
8. $18x^2 - 9x - 2$.
9. $8x^2 - 6x + 1$.
10. $x^2 - (2a - c)x - 2ac$.

385. SYNOPSIS FOR REVIEW.

CHAPTER XIII.
EQUATIONS OF THE SECOND DEGREE WITH ONE UNKNOWN QUANTITY.

- Equations of the Second Degree with only one Unknown Quantity.
- General form of Equation of Second Degree with only one Unknown Quantity.
- Quadratic Equations.
- Complete and Incomplete Equations.
- Solution of Incomplete Equation and Rule. Cor.
- Solution of Complete Equation and Rule.
- Application of the formulæ:
  - $x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}$.
  - $x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$.
- Hindoo method of completing the square
- Method of causing the term containing the first power of the unknown quantity to disappear.
- Propositions relating to Quadratic Equations:
  - *Number of roots.*
  - *Sum of roots.*
  - *Product of roots.* Cor.
- Discussion of the equation $x^2 + px = q$:
  - *Signs of the arbitrary quantities.*
  - *The four forms and corresponding roots.*
  - *Unequal roots.*
  - *Equal roots.*
  - *Real roots.*
  - *Imaginary roots.*
- Problem of Lights.
- Quadratic Expressions . . .
  - *Difference between quadratic expression and quadratic equation.*
  - *To resolve into factors.*

CHAPTER XIV.

HIGHER EQUATIONS

WITH ONE UNKNOWN QUANTITY.

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386. There are many equations which, though not really of the second degree, may be solved by processes similar to those given in the preceding chapter. To this class belong: 1st. All equations of the higher degrees which contain only one power of the unknown quantity; 2d. All equations of the higher degrees which contain two, and only two, powers of the unknown quantity, and in which the exponent of one of these powers is double that of the other. Such equations may be reduced to one or the other of the forms

$$ax^n = c \quad . \quad . \quad . \quad (1),$$

$$ax^{2n} + bx^n = c \quad . \quad . \quad . \quad (2).$$

Equation (1) is called a *pure* equation of the n^{th} degree. Equation (2) is said to have the *form* of a complete equation of the second degree.

EXAMPLES.

1. Solve the equation $ax^n = c$.

Dividing both members by a, and extracting the n^{th} root of both members of the resulting equation, we find

$$x = \sqrt[n]{\frac{c}{a}}.$$

2. Solve the equation $ax^{2n} + bx^n = c$.

By division, $\qquad x^{2n} + \frac{b}{a}x^n = \frac{c}{a};$

by completing the square,

$$x^{2n}+\frac{b}{a}x^n+\left(\frac{b}{2a}\right)^2=\frac{c}{a}+\left(\frac{b}{2a}\right)^2=\frac{4ac+b^2}{4a^2};$$

by extracting the square root,

$$x^n+\frac{b}{2a}=\pm\frac{\sqrt{4ac+b^2}}{2a};$$

by transposition, $$x^n=\frac{-b\pm\sqrt{4ac+b^2}}{2a};$$

by extracting the n^{th} root,

$$x=\sqrt[n]{\frac{-b\pm\sqrt{4ac+b^2}}{2a}}.$$

3. Solve the equation $x^4-9x^2=-20$.

Completing the square,

$$x^4-9x^2+\left(\frac{9}{2}\right)^2=-20+\frac{81}{4}=\frac{1}{4};$$

whence, $$x^2-\frac{9}{2}=\pm\frac{1}{2};$$

$\therefore$ $$x^2=\frac{9}{2}\pm\frac{1}{2}=5 \text{ or } 4,$$

and $$x=\pm\sqrt{5} \text{ or } \pm 2.$$

4. Solve the equation $x^2-7x+\sqrt{x^2-7x+18}=24$.

Adding 18 to both members,

$$x^2-7x+18+\sqrt{x^2-7x+18}=42.$$

Assuming $\sqrt{x^2-7x+18}=y$, this equation becomes

$$y^2+y=42;$$

whence, $$y=6 \text{ or } -7.$$

We have now the two equations

$$\sqrt{x^2-7x+18}=6,$$
$$\sqrt{x^2-7x+18}=-7,$$

from the first of which we find $x=9$ or -2, and from the second, $x=\frac{1}{2}(7\pm\sqrt{173})$.

5. Solve the equation

$$x^4 + 6x^3 + 80x^2 + 213x - 2128 = 0 \quad . \quad . \quad . \quad (1).$$

We seek to transform (1) into another equation such that its first three terms shall be the square of a binomial, and the remaining terms shall contain the first power of that binomial. We see that $x^4 + 6x^3$ contains two terms of the square of $x^2 + 3x$, and that we only need to add $9x^2$ to it to complete the square. Separating the term $80x^2$ into the two parts, $9x^2$ and $71x^2$, (1) may be written thus,

$$x^4 + 6x^3 + 9x^2 + 71x^2 + 213x = 2128,$$

or, $$(x^2 + 3x)^2 + 71\,(x^2 + 3x) = 2128 \quad . \quad . \quad . \quad (2).$$

Assuming $x^2 + 3x = y$, (2) becomes

$$y^2 + 71y = 2128 \quad . \quad . \quad . \quad (3);$$

whence, $$y = \frac{-71 \pm \sqrt{13553}}{2},$$

We have now the two equations

$$x^2 + 3x = \frac{-71 + \sqrt{13553}}{2},$$

$$x^2 + 3x = \frac{-71 - \sqrt{13553}}{2},$$

which are easily solved.

In the answers to some of the following examples some of the roots are omitted.

6. Solve $3x^3 + 42x^{\frac{3}{2}} = 3321$. *Ans.* $x = 9$ or $(-41)^{\frac{2}{3}}$.

7. Solve $x^{10} + 31x^5 = 32$. *Ans.* $x = 1$ or -2.

8. Solve $x^6 - 35x^3 + 216 = 0$. *Ans.* $x = 2$ or 3.

9. Solve $x^{\frac{1}{n}} - x^{\frac{2}{n}} + 2 = 0$. *Ans.* $x = 2^n$ or $(-1)^n$.

10. Solve $x^{\frac{1}{n}} - 13x^{\frac{1}{2n}} = 14$. *Ans.* $x = 14^{2n}$ or $(-1)^{2n}$.

11. Solve $x^{\frac{1}{3}} + \dfrac{5}{2x^{\frac{1}{3}}} = 3\frac{1}{4}$. *Ans.* $x = 8$ or $\dfrac{125}{64}$.

12. Solve $x^4 - 14x^2 + 40 = 0$. *Ans.* $x = \pm 2$ or $\pm\sqrt{10}$.

13. Solve $2\left(x^{\frac{1}{n}} + x^{-\frac{1}{n}}\right) = 5$. *Ans.* $x = 2^n$ or $\dfrac{1}{2^n}$.

14. Solve $\left(\dfrac{x}{x-1}\right)^2 + \left(\dfrac{x}{x+1}\right)^2 = n(n-1)$.

Ans. $x = \pm\sqrt{\dfrac{n}{n-2}}$ or $\pm\sqrt{\dfrac{n-1}{n+1}}$.

15. Solve $x^2 + 5x + 4 = 5\sqrt{x^2 + 5x + 28}$.

Ans. $x = 4$ or -9, or $-\dfrac{5}{2} \pm \dfrac{1}{2}\sqrt{-51}$.

387. ***PROBLEMS.***

1. A vintner draws a certain quantity of wine out of a full cask that holds 256 gallons, and then, filling the cask with water, draws out the same quantity of liquor as before, and so on for four draughts, when there were only 81 gallons of wine left. Supposing the water and wine to become thoroughly mixed every time the cask is filled, how much wine did he draw off the first time?

Ans. 64 gallons.

2. A number a is diminished by the x^{th} part of itself; the remainder thus obtained is diminished by the x^{th} part of itself; and so on to the fourth remainder, which is equal to b. Find the value of x.

Ans. $x = \dfrac{\sqrt[4]{a}}{\sqrt[4]{a} - \sqrt[4]{b}}$.

3. Find two numbers whose sum is 8, and the sum of whose fourth powers is 706. *Ans.* 3 and 5.

4. Find two numbers whose sum is $2a$, and the sum of whose fourth powers is $2b$.

Ans. $a + \sqrt{\sqrt{8a^4 + b} - 3a^2}$ and $a - \sqrt{\sqrt{8a^4 + b} - 3a^2}$.

CHAPTER XV.

SIMULTANEOUS EQUATIONS.

DEFINITIONS.

388. The particular case in which all the equations of a group are of the first degree has been considered in Chapter VII. We propose, in the present chapter, to consider groups, each of which contains at least one equation of a higher degree than the first.

A group containing only two equations is sometimes called a ***Pair.***

389. When the members of an equation are *entire* and *rational* with reference to its unknown quantities, the ***Degree*** of the equation is the sum of the exponents of the unknown quantities in the term where this sum is the greatest. Thus,

$$4xy - 3x = 2 - 5y$$

is an equation of the *second* degree.

390. If an equation of the second degree involving only two unknown quantities, x and y, contains all the kinds of terms of which it is susceptible, it can be reduced to the form of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

PAIRS OF EQUATIONS ONE OF WHICH IS OF THE FIRST AND THE OTHER OF THE SECOND DEGREE.

391. ***EXAMPLES.***

1. Solve the equations

$$x + y = 6 \quad . \quad . \quad . \quad (1),$$

$$x^2 + 3xy + y^2 = 44 \quad . \quad . \quad . \quad (2).$$

From (1), $y = 6 - x$.

Substituting this value for y in (2),

$$x^2 + 3x(6 - x) + (6 - x)^2 = 44;$$

whence, $x = 4$ or 2.

Substituting these values for x in the equation

$$y = 6 - x,$$

we find $y = 2$ or 4.

2. Solve the equations

$$ax + by = c \quad . \quad . \quad . \quad (1),$$

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad . \quad . \quad . \quad (2).$$

From (1), $y = \dfrac{c - ax}{b}$.

Substituting this value for y in (2),

$$Ax^2 + Bx\left(\frac{c - ax}{b}\right) + C\left(\frac{c - ax}{b}\right)^2 + Dx + E\left(\frac{c - ax}{b}\right) + F = 0.$$

Clearing this equation of fractions and collecting similar terms,

$$(Ab^2 - Bab + Ca^2)\,x^2 + (Bbc - 2Cac + Db^2 - Eab)\,x + Cc^2 + Ebc + Fb^2 = 0 \quad . \quad . \quad . \quad (3).$$

Representing each of the coefficients in this equation by a single letter, it may be written thus:

$$mx^2 + nx + p = 0 \quad . \quad . \quad . \quad (4).$$

This equation will furnish two values for x; then substituting these in the equation $y = \dfrac{c - ax}{b}$, we shall have the corresponding values of y.

3. Solve the equations

$$x + y = a \quad . \quad . \quad . \quad (1),$$

$$xy = b^2 \quad . \quad . \quad . \quad (2).$$

From (2), $$y = \frac{b^2}{x}.$$

Substituting this value for y in (1),

$$x + \frac{b^2}{x} = a \quad . \quad . \quad . \quad (3);$$

whence, $$x = \frac{a \pm \sqrt{a^2 - 4b^2}}{2}.$$

Substituting these values for x in (1), we find

$$y = \frac{a \mp \sqrt{a^2 - 4b^2}}{2}$$

Another Solution.—The values of x and y are the roots of the equation

$$z^2 - az = -b^2 \quad \mathbf{(365\text{-}366)};$$

$$\therefore \qquad x \text{ or } y = \frac{a \pm \sqrt{a^2 - 4b^2}}{2}.$$

4. Solve the equations

$$x - y = a \quad . \quad . \quad . \quad (1),$$

$$xy = b^2 \quad . \quad . \quad . \quad (2).$$

Eliminating y,

$$x - \frac{b^2}{x} = a \quad . \quad . \quad . \quad (3);$$

whence, $$x = \frac{a \pm \sqrt{a^2 + 4b^2}}{2}.$$

The corresponding values of y are

$$\frac{-a \pm \sqrt{a^2 + 4b^2}}{2}.$$

Another Solution.—Put $y = -v$; then (1) and (2) become

$$x + v = a,$$

$$xv = -b^2.$$

Hence the values of x and v are the roots of the equation

$$z^2 - az = b^2;$$

$$\therefore \qquad x \text{ or } v = \frac{a \pm \sqrt{a^2 + 4b^2}}{2};$$

that is,

$$x = \frac{a \pm \sqrt{a^2 + 4b^2}}{2},$$

$$y = \frac{-a \pm \sqrt{a^2 + 4b^2}}{2}.$$

5. Solve the equations

$$x + y = a \quad . \quad . \quad . \quad (1),$$

$$x^2 + y^2 = b^2 \quad . \quad . \quad . \quad (2).$$

Squaring both members of (1),

$$x^2 + 2xy + y^2 = a^2 \quad . \quad . \quad . \quad (3).$$

Subtracting (2) from (3),

$$2xy = a^2 - b^2 \quad . \quad . \quad . \quad (4).$$

Subtracting (4) from (2),

$$x^2 - 2xy + y^2 = 2b^2 - a^2 \quad . \quad . \quad . \quad (5);$$

whence,

$$x - y = \pm\sqrt{2b^2 - a^2} \quad . \quad . \quad . \quad (6).$$

Combining (1) and (6), we find

$$x = \frac{a \pm \sqrt{2b^2 - a^2}}{2},$$

$$y = \frac{a \mp \sqrt{2b^2 - a^2}}{2}.$$

Solve the following pairs of equations:

6. $\begin{cases} x^2 - 2y^2 = 71 \\ x + y = 20 \end{cases}$. *Ans.* $\begin{cases} x = 67 \text{ or } 13, \\ y = -47 \text{ or } 7. \end{cases}$

7. $\begin{cases} 2x^2 + xy - 5y^2 = 20 \\ 2x - 3y = 1 \end{cases}$. *Ans.* $\begin{cases} x = 5 \text{ or } -\dfrac{37}{4}, \\ y = 3 \text{ or } -\dfrac{13}{2}. \end{cases}$

8. $\begin{Bmatrix} x + y = 100 \\ xy = 2400 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 60 \text{ or } 40, \\ y = 40 \text{ or } 60. \end{cases}$

9. $\begin{Bmatrix} x + y = 4 \\ \frac{1}{x} + \frac{1}{y} = 1 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 2, \\ y = 2. \end{cases}$

10. $\begin{Bmatrix} x + y = 7 \\ x^2 + 2y^2 = 34 \end{Bmatrix}$. *Ans.* $\begin{cases} x = \frac{16}{3} \text{ or } 4, \\ y = \frac{5}{3} \text{ or } 3. \end{cases}$

11. $\begin{Bmatrix} x - y = 12 \\ x^2 + y^2 = 74 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 7 \text{ or } 5, \\ y = -5 \text{ or } -7. \end{cases}$

12. $\begin{Bmatrix} x - \frac{x-y}{2} = 4 \\ y - \frac{x+3y}{x+2} = 1 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 5 \text{ or } 2, \\ y = 3 \text{ or } 6. \end{cases}$

13. $\begin{Bmatrix} 3x - 5y = 2 \\ xy = 1 \end{Bmatrix}$. *Ans.* $\begin{cases} x = \frac{5}{3} \text{ or } -1, \\ y = \frac{3}{5} \text{ or } -1. \end{cases}$

14. $\begin{Bmatrix} \frac{x}{a} + \frac{y}{b} = 1 \\ \frac{a}{x} + \frac{b}{y} = 4 \end{Bmatrix}$. *Ans.* $\begin{cases} x = \frac{a}{2}, \\ y = \frac{b}{2}. \end{cases}$

15. $\begin{Bmatrix} \frac{a}{a+x} + \frac{b}{b+y} = 1 \\ x + y = a + b \end{Bmatrix}$. *Ans.* $\begin{cases} x = a \text{ or } b, \\ y = b \text{ or } a. \end{cases}$

16. $\begin{Bmatrix} \frac{1}{x^{-1}} + \frac{1}{y^{-1}} = 8 \\ \frac{1}{x^{-2}} + \frac{1}{y^{-2}} = 34 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 5 \text{ or } 3, \\ y = 3 \text{ or } 5. \end{cases}$

17. $\begin{Bmatrix} \frac{1}{x^{-1}} + \frac{4}{y^{-1}} = 14 \\ \frac{1}{y^{-2}} + \frac{4}{x^{-1}} = \frac{2}{y^{-1}} + 11 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 2 \text{ or } -46, \\ y = 3 \text{ or } 15. \end{cases}$

18. $\left\{\begin{array}{c} x - y = 2 \\ xy^{-1} - yx^{-1} = \dfrac{16}{15} \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 5 \text{ or } \dfrac{3}{4}, \\ y = 3 \text{ or } -\dfrac{5}{4}. \end{array}\right.$

19. $\left\{\begin{array}{c} 2(x+y)^{-1}+3(x-y)^{-1}=\dfrac{11}{3} \\ x^0 + 2x - y = 4 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=2 \text{ or } \dfrac{15}{11}, \\ y=1 \text{ or } -\dfrac{3}{11}. \end{array}\right.$

PARTICULAR SYSTEMS.

392. If one of the given equations is of a higher degree than the second, or if both are of a higher degree than the first, the elimination of one of the unknown quantities usually leads to an equation of a higher degree than the second.

Illustrations.—If we eliminate y from the equations

$$\left\{\begin{array}{c} ax + by = c \\ x^3 + x^2y + xy + y = d \end{array}\right\},$$

we obtain

$$(b-a)x^3 + (c-a)x^2 + (c-a)x = bd - c;$$

and if we eliminate y from the equations

$$\left\{\begin{array}{c} x^2 + y^2 = 13 \\ x + xy + y = 11 \end{array}\right\},$$

we obtain

$$x^4 + 2x^3 - 11x^2 - 48x = -108.$$

Since we have thus far had no *general* method for the solution of equations of a higher degree than the second, it follows that we cannot now give a *general* rule for the solution of pairs of simultaneous equations where one of them is of a higher degree than the second, or where both are of a higher degree than the first. We, however, frequently meet with pairs of this kind that can be solved by the aid of special artifices.

EXAMPLES.

1. Solve the equations

$$x^2 + y^2 = 25 \quad . \quad . \quad . \quad (1),$$

$$xy = 12 \quad . \quad . \quad . \quad (2).$$

Multiplying (2) by 2 and adding the result to (1),

$$(x + y)^2 = 49;$$

whence, $$x + y = \pm 7 \quad . \quad . \quad . \quad (3).$$

Subtracting the equation $2xy = 24$ from (1),

$$(x - y)^2 = 1;$$

whence, $$x - y = \pm 1 \quad . \quad . \quad . \quad (4).$$

We have now four groups to consider, namely:

$$\begin{Bmatrix} x + y = 7 \\ x - y = 1 \end{Bmatrix}; \qquad \begin{Bmatrix} x + y = 7 \\ x - y = -1 \end{Bmatrix};$$

$$\begin{Bmatrix} x + y = -7 \\ x - y = 1 \end{Bmatrix}; \qquad \begin{Bmatrix} x + y = -7 \\ x - y = -1 \end{Bmatrix}.$$

Solving these four groups, we obtain

$$\begin{Bmatrix} x = \pm 3 \\ y = \pm 4 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} x = \pm 4 \\ y = \pm 3 \end{Bmatrix}.$$

2. Solve the equations

$$x^2 + y^2 = a^2 \quad . \quad . \quad . \quad (1),$$

$$xy = b^2 \quad . \quad . \quad . \quad (2).$$

Multiplying (2) by 2 and adding the result to (1),

$$(x + y)^2 = a^2 + 2b^2;$$

whence, $$x + y = \pm \sqrt{a^2 + 2b^2} \quad . \quad . \quad . \quad (3).$$

Subtracting the equation $2xy = 2b^2$ from (1),

$$(x - y)^2 = a^2 - 2b^2;$$

whence, $$x - y = \pm \sqrt{a^2 - 2b^2} \quad . \quad . \quad . \quad (4).$$

Adding (3) and (4),

$$2x = \pm\sqrt{a^2 + 2b^2} \pm \sqrt{a^2 - 2b^2};$$

whence, $$x = \pm\tfrac{1}{2}\sqrt{a^2 + 2b^2} \pm \tfrac{1}{2}\sqrt{a^2 - 2b^2}.$$

Subtracting (4) from (3),

$$2y = \pm\sqrt{a^2 + 2b^2} \mp \sqrt{a^2 - 2b^2};$$

whence, $$y = \pm\tfrac{1}{2}\sqrt{a^2 + 2b^2} \mp \tfrac{1}{2}\sqrt{a^2 - 2b^2}.$$

3. Solve the equations

$$x^2 - y^2 = a^2 \quad . \quad . \quad . \quad (1),$$

$$xy = b^2 \quad . \quad . \quad . \quad (2).$$

Adding four times the square of (2) to the square of (1),

$$x^4 + 2x^2y^2 + y^4 = a^4 + 4b^4 \quad . \quad . \quad . \quad (3);$$

whence, $$x^2 + y^2 = \pm\sqrt{a^4 + 4b^4} \quad . \quad . \quad . \quad (4).$$

Adding (1) to (4),

$$2x^2 = a^2 \pm \sqrt{a^4 + 4b^4};$$

whence, $$x = \pm\sqrt{\frac{a^2 \pm \sqrt{a^4 + 4b^4}}{2}}.$$

Subtracting (1) from (4),

$$2y^2 = -a^2 \pm \sqrt{a^4 + 4b^4};$$

whence, $$y = \pm\sqrt{\frac{-a^2 \pm \sqrt{a^4 + 4b^4}}{2}}.$$

4. Solve the equations

$$y^2 - x^2 = 16 \quad . \quad . \quad . \quad (1),$$

$$2y^2 - 4xy + 3x^2 = 17 \quad . \quad . \quad . \quad (2).$$

Assume $y = vx$; then (1) and (2) become

$$v^2x^2 - x^2 = 16 \quad . \quad . \quad . \quad (3),$$

$$2v^2x^2 - 4vx^2 + 3x^2 = 17 \quad . \quad . \quad . \quad (4).$$

From (3), $$x^2 = \frac{16}{v^2 - 1},$$

and from (4), $$x^2 = \frac{17}{2v^2 - 4v + 3};$$

$$\therefore \qquad \frac{17}{2v^2 - 4v + 3} = \frac{16}{v^2 - 1} \quad . \quad . \quad . \quad (5).$$

Clearing (5) of fractions, transposing, and reducing,

$$15v^2 - 64v = -65 \quad . \quad . \quad . \quad (6);$$

whence, $$v = \frac{5}{3} \text{ or } \frac{13}{5}.$$

Substituting $\frac{5}{3}$ for v in the two equations $x^2 = \frac{16}{v^2 - 1}$ and $y = vx$, we have $x^2 = 9$, and $y = \frac{5}{3}x$;

$$\therefore \qquad x = \pm 3 \quad \text{and} \quad y = \pm 5.$$

Substituting $\frac{13}{5}$ for v in the same equations, we find

$$x = \pm \frac{5}{3} \quad \text{and} \quad y = \pm \frac{13}{3}.$$

The artifice here used may be adopted whenever all the unknown terms in both equations are of the second degree with reference to the unknown quantities.

5. Solve the equations

$$x^2 + xy - 6y^2 = 24 \quad . \quad . \quad . \quad (1),$$

$$x^2 + 3xy - 10y^2 = 32 \quad . \quad . \quad . \quad (2).$$

Assuming $y = vx$, (1) and (2) become

$$x^2 + vx^2 - 6v^2x^2 = 24 \quad . \quad . \quad . \quad (3),$$

$$x^2 + 3vx^2 - 10v^2x^2 = 32 \quad . \quad . \quad . \quad (4).$$

From (3), $$x^2 = \frac{24}{1+v-6v^2},$$

and from (4), $$x^2 = \frac{32}{1+3v-10v^2};$$

$$\therefore \quad \frac{24}{1+v-6v^2} = \frac{32}{1+3v-10v^2} \quad . \quad . \quad . \quad (5);$$

whence, $$v = \frac{1}{2} \text{ or } \frac{1}{3}.$$

Substituting $\frac{1}{2}$ for v in the two equations $x^2 = \dfrac{24}{1+v-6v^2}$ and $y = vx$, we have $x^2 = \infty$, and $y = \frac{1}{2}x$;

$$\therefore \quad x = \pm\infty \quad \text{and} \quad y = \pm\infty.$$

Substituting $\frac{1}{3}$ for v in the same equations,

$$x = \pm 6 \quad \text{and} \quad y = \pm 2.$$

6. Solve the equations

$$x^2 + 2xy + y^2 + ax + ay = b \quad . \quad . \quad . \quad (1),$$
$$xy + y^2 = c \quad . \quad . \quad . \quad (2).$$

The first equation may be written thus:

$$(x+y)^2 + a(x+y) = b;$$

whence, $$x + y = \frac{-a \pm \sqrt{a^2+4b}}{2} \quad . \quad . \quad . \quad (3).$$

From (2), $$x + y = \frac{c}{y} \quad . \quad . \quad . \quad (4).$$

Combining (3) and (4), we find

$$x = \frac{a^2+2b-2c \mp a\sqrt{a^2+4b}}{-a \pm \sqrt{a^2+4b}},$$

$$y = \frac{2c}{-a \pm \sqrt{a^2+4b}}.$$

7. Solve the equations

$$x^2 - 2xy + y^2 - ax + ay = b \quad . \quad . \quad . \quad (1),$$

$$xy - y^2 = c \quad . \quad . \quad . \quad (2).$$

Equation (1) may be written thus:

$$(x - y)^2 - a(x - y) = b;$$

whence, $$x - y = \frac{a \pm \sqrt{a^2 + 4b}}{2} \quad . \quad . \quad . \quad (3).$$

From (2), $$x - y = \frac{c}{y} \quad . \quad . \quad . \quad (4).$$

Combining (3) and (4), we find

$$x = \frac{a^2 + 2b + 2c \pm a\sqrt{a^2 + 4b}}{a \pm \sqrt{a^2 + 4b}},$$

$$y = \frac{2c}{a \pm \sqrt{a^2 + 4b}}.$$

8. Solve the equations

$$x^2 y + xy^2 = 30 \quad . \quad . \quad . \quad (1),$$

$$\frac{1}{x} + \frac{1}{y} = \frac{5}{6} \quad . \quad . \quad . \quad (2).$$

Equation (1) may be reduced to the form

$$xy(x + y) = 30 \quad . \quad . \quad . \quad (3),$$

and (2) may be reduced to the form

$$6(x + y) = 5xy \quad . \quad . \quad . \quad (4).$$

Dividing (3) by (4),

$$\frac{xy}{6} = \frac{30}{5xy};$$

whence, $$5x^2y^2 = 180,$$

or, $$x^2y^2 = 36;$$

whence, $$xy = \pm 6 \quad . \quad . \quad . \quad (5).$$

Combining (3) and (5),

$$x+y=\pm 5 \quad . \quad . \quad . \quad (6).$$

Combining (5) and (6), we find

$$x=3,\ 2,\quad 1,\ \text{ or }\ -6,$$
$$y=2,\ 3,\ -6,\ \text{ or }\quad 1.$$

9. Solve the equations

$$x+y=a \quad . \quad . \quad . \quad (1),$$
$$x^5+y^5=b^5 \quad . \quad . \quad . \quad (2).$$

Dividing (2) by (1),

$$x^4-x^3y+x^2y^2-xy^3+y^4=\frac{b^5}{a};$$

which may be placed under the form

$$x^4+y^4-xy\,(x^2+y^2)+x^2y^2=\frac{b^5}{a} \quad . \quad . \quad . \quad (3).$$

Squaring (1) and transposing the term $2xy$,

$$x^2+y^2=a^2-2xy \quad . \quad . \quad . \quad (4).$$

Squaring (4),

$$x^4+2x^2y^2+y^4=a^4-4a^2xy+4x^2y^2 \quad . \quad . \quad . \quad (5);$$

whence, $\quad x^4+y^4=a^4-4a^2xy+2x^2y^2 \quad . \quad . \quad . \quad (6).$

Equation (3) may therefore be placed under the form

$$a^4-4a^2xy+2x^2y^2-xy\,(a^2-2xy)+x^2y^2=\frac{b^5}{a};$$

that is, $\quad 5x^2y^2-5a^2xy=\dfrac{b^5}{a}-a^4 \quad . \quad . \quad . \quad (7);$

whence, $\quad xy=\dfrac{a^2\pm\sqrt{\dfrac{4b^5+a^5}{5a}}}{2} \quad . \quad . \quad . \quad (8).$

The values of x and y may now be found by combining (1) and (8).

Solve the following pairs of equations:

10. $\left\{\begin{aligned} x^2+y^2&=65 \\ xy&=28 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x&=\pm 7 \text{ or } \pm 4, \\ y&=\pm 4 \text{ or } \pm 7. \end{aligned}\right.$

11. $\left\{\begin{aligned} x^2+xy+2y^2&=74 \\ 2x^2+2xy+y^2&=73 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x&=\pm 3 \text{ or } \mp 8, \\ y&=\pm 5. \end{aligned}\right.$

12. $\left\{\begin{aligned} x^2+3xy&=54 \\ xy+4y^2&=115 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x&=\pm 3 \text{ or } \pm 36, \\ y&=\pm 5 \text{ or } \mp \frac{23}{2}. \end{aligned}\right.$

13. $\left\{\begin{aligned} x^2+xy&=15 \\ xy-y^2&=2 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x&=\pm 3 \text{ or } \pm \frac{5}{\sqrt{2}}, \\ y&=\pm 2 \text{ or } \pm \frac{1}{\sqrt{2}}. \end{aligned}\right.$

14. $\left\{\begin{aligned} x^2+xy+4y^2&=6 \\ 3x^2+8y^2&=14 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x&=\pm 2 \text{ or } \pm\sqrt{\frac{2}{5}}, \\ y&=\pm \frac{1}{2} \text{ or } \mp 2\sqrt{\frac{2}{5}}. \end{aligned}\right.$

15. $\left\{\begin{aligned} x^2+xy&=12 \\ xy-2y^2&=1 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x&=\pm 3 \text{ or } \pm \frac{8}{\sqrt{6}}, \\ y&=\pm 1 \text{ or } \pm \frac{1}{\sqrt{6}}. \end{aligned}\right.$

16. $\left\{\begin{aligned} x^2-xy+y^2&=21 \\ y^2-2xy+15&=0 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x&=\pm 4 \text{ or } \pm 3\sqrt{3}, \\ y&=\pm 5 \text{ or } \pm \sqrt{3}. \end{aligned}\right.$

17. $\left\{\begin{aligned} x^2-4y^2&=9 \\ xy+2y^2&=3 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x&=\pm \frac{15}{\sqrt{21}} \text{ or } \pm\infty, \\ y&=\pm \frac{3}{\sqrt{21}} \text{ or } \mp\infty. \end{aligned}\right.$

18. $\left\{\begin{aligned} &\frac{1}{x}+\frac{1}{y}=\frac{1}{5} \\ &xy=180 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x&=30 \text{ or } 6, \\ y&=6 \text{ or } 30. \end{aligned}\right.$

19. $\left\{\begin{array}{l}\dfrac{3(x+y)}{x-y}+\dfrac{3(x-y)}{x+y}=10 \\ x^2+y^2=45\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 6, \\ y=\pm 3 \text{ or } \mp 3.\end{array}\right.$

20. $\left\{\begin{array}{l}x+y=x^2 \\ 3y-x=y^2\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=0, 2, \text{ or } \pm\sqrt{2}, \\ y=0, 2, \text{ or } 2\mp\sqrt{2}.\end{array}\right.$

21. $\left\{\begin{array}{l}x^2+y^2=\frac{5}{2}xy \\ x-y=\frac{1}{4}xy\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=0, 4, \text{ or } -2, \\ y=0, 2, \text{ or } -4.\end{array}\right.$

22. $\left\{\begin{array}{l}x^2+y^2+x+y=18 \\ xy=6\end{array}\right\}$.

Ans. $\left\{\begin{array}{l}x=3, 2, \text{ or } -3\pm\sqrt{3}, \\ y=2, 3, \text{ or } -3\mp\sqrt{3}.\end{array}\right.$

23. $\left\{\begin{array}{l}x^2+y^2-x-y=32 \\ x+y+xy=29\end{array}\right\}$.

Ans. $\left\{\begin{array}{l}x=5, 4, \text{ or } \dfrac{-10\pm\sqrt{-56}}{2}, \\ y=4, 5, \text{ or } \dfrac{-10\mp\sqrt{-56}}{2}.\end{array}\right.$

24. $\left\{\begin{array}{l}x+y=5 \\ x^3+y^3=65\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=4 \text{ or } 1, \\ y=1 \text{ or } 4.\end{array}\right.$

25. $\left\{\begin{array}{l}x-y=2 \\ x^3-y^3=8\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=2 \text{ or } 0, \\ y=0 \text{ or } -2.\end{array}\right.$

In some of the following answers the roots are not all given:

26. $\left\{\begin{array}{l}xy(x+y)=30 \\ x^3+y^3=35\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=3 \text{ or } 2, \\ y=2 \text{ or } 3.\end{array}\right.$

27. $\left\{\begin{array}{l}x+y=4 \\ x^4+y^4=82\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=3, 1, \text{ or } 2\pm 5\sqrt{-1}, \\ y=1, 3, \text{ or } 2\mp 5\sqrt{-1}.\end{array}\right.$

28. $\left\{\begin{array}{l}x^5-y^5=3093 \\ x-y=3\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=5 \text{ or } -2, \\ y=2 \text{ or } -5.\end{array}\right.$

29. $\left\{\begin{matrix} x^2 - x^2y^2 + y^2 = 19 \\ x - xy + y = 4 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x = \frac{1}{4}(9 \pm \sqrt{73}), \\ y = \frac{1}{4}(9 \mp \sqrt{73}). \end{matrix}\right.$

30. $\left\{\begin{matrix} x^2 - xy + y^2 = 7 \\ x^4 + x^2y^2 + y^4 = 133 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x = \pm 3 \text{ or } \pm 2, \\ y = \pm 2 \text{ or } \pm 3. \end{matrix}\right.$

31. $\left\{\begin{matrix} x^2 + y^2 + xy = 49 \\ x^4 + y^4 + x^2y^2 = 931 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x = \pm 5 \text{ or } \pm 3, \\ y = \pm 3 \text{ or } \pm 5. \end{matrix}\right.$

32. $\left\{\begin{matrix} x^4 - x^2 + y^4 - y^2 = 84 \\ x^2 + x^2y^2 + y^2 = 49 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x = \pm 3 \text{ or } \pm 2, \\ y = \pm 2 \text{ or } \pm 3. \end{matrix}\right.$

33. $\left\{\begin{matrix} (x^2 - y^2)(x - y) = 16xy \\ (x^4 - y^4)(x^2 - y^2) = 640x^2y^2 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x = 9 \text{ or } 3, \\ y = 3 \text{ or } 9. \end{matrix}\right.$

34. $\left\{\begin{matrix} ax = y(s - y) \\ y^2 + (s - y)^2 = x^2 \end{matrix}\right\}$.

Ans. $\left\{\begin{matrix} x = -a \pm \sqrt{s^2 + a^2}, \\ y = \dfrac{s \pm \sqrt{4a^2 + s^2 \mp 4a\sqrt{s^2 + a^2}}}{2}. \end{matrix}\right.$

35. $\left\{\begin{matrix} x^2y^{-1} + y^2x^{-1} = 9 \\ x + y = 6 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x = 4 \text{ or } 2, \\ y = 2 \text{ or } 4. \end{matrix}\right.$

PAIRS OF EQUATIONS INVOLVING RADICAL QUANTITIES.

393. *EXAMPLES.*

1. Solve the equations

$$\sqrt{x} + \sqrt{y} = 5 \quad . \quad . \quad . \quad (1),$$

$$x + y = 13 \quad . \quad . \quad . \quad (2).$$

Squaring (1) and subtracting (2) from the result,

$$2\sqrt{xy} = 12;$$

whence, $$4xy = 144 \quad . \quad . \quad . \quad (3).$$

Squaring (2) and subtracting (3) from the result,

$$x^2 - 2xy + y^2 = 25;$$

whence, $$x - y = \pm 5 \quad . \quad . \quad . \quad (4).$$

Combining (2) and (4), we find $\begin{cases} x = 9 \text{ or } 4, \\ y = 4 \text{ or } 9. \end{cases}$

2. Solve the equations

$$x - y + \sqrt{\frac{x-y}{x+y}} = \frac{20}{x+y} \quad . \quad . \quad . \quad (1),$$

$$x^2 + y^2 = 34 \quad . \quad . \quad . \quad (2).$$

From (1), by transposition and reduction,

$$\frac{x^2 - y^2 - 20}{x+y} = -\sqrt{\frac{x-y}{x+y}} \quad . \quad . \quad . \quad (3).$$

Dividing the denominators of (3) by $\sqrt{x+y}$,

$$\frac{x^2 - y^2 - 20}{\sqrt{x+y}} = -\sqrt{x-y};$$

whence, $$x^2 - y^2 - 20 = -\sqrt{x^2 - y^2};$$

or, $$x^2 - y^2 + \sqrt{x^2 - y^2} = 20 \quad . \quad . \quad . \quad (4).$$

Assuming $\sqrt{x^2 - y^2} = z$, (4) becomes

$$z^2 + z = 20 \quad . \quad . \quad . \quad (5);$$

whence, $$z = 4 \text{ or } -5;$$

that is, $$x^2 - y^2 = 16 \text{ or } 25 \quad . \quad . \quad . \quad (6).$$

Combining (2) and (6), we find $\begin{cases} x = \pm 5 \text{ or } \pm\sqrt{\dfrac{59}{2}}, \\ y = \pm 3 \text{ or } \pm\dfrac{3}{\sqrt{2}}. \end{cases}$

3. Solve the equations

$$x^{\frac{3}{4}} + y^{\frac{3}{5}} = 35 \quad . \quad . \quad . \quad (1),$$

$$x^{\frac{1}{4}} + y^{\frac{1}{5}} = 5 \quad . \quad . \quad . \quad (2).$$

Assuming $x^{\frac{1}{4}} = v$ and $y^{\frac{1}{5}} = z$, (1) and (2) become

$$v^3 + z^3 = 35 \quad . \quad . \quad . \quad (3),$$

$$v + z = 5 \quad . \quad . \quad . \quad (4).$$

These equations may now be solved as Ex. 24, Art. **392.**

4. Solve the equations

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{61}{\sqrt{xy}} + 1 \quad . \quad . \quad . \quad (1),$$

$$\sqrt[4]{x^3y} + \sqrt[4]{xy^3} = 78 \quad . \quad . \quad . \quad (2).$$

Clearing (1) of fractions, it becomes

$$x + y = 61 + \sqrt{xy} \quad . \quad . \quad . \quad (3).$$

Equation (2) may be written

$$\sqrt[4]{xy}(\sqrt{x} + \sqrt{y}) = 78 \quad . \quad . \quad . \quad (4).$$

Assume $$x + y = v \quad . \quad . \quad . \quad (5),$$

and $$\sqrt{xy} = z \quad . \quad . \quad . \quad (6).$$

Multiplying (6) by 2 and adding the result to (5),

$$x + 2\sqrt{xy} + y = v + 2z;$$

whence, $$\sqrt{x} + \sqrt{y} = \pm\sqrt{v + 2z} \quad . \quad . \quad . \quad (7).$$

Hence (3) and (4) become

$$v = 61 + z \quad . \quad . \quad . \quad (8),$$

$$\sqrt{z}(\pm\sqrt{v + 2z}) = 78 \quad . \quad . \quad . \quad (9).$$

The values of v and z are easily found from (8) and (9), and then (5) and (6) will furnish the values of x and y.

Solve the following pairs of equations:

5. $\left.\begin{cases} x+y+\sqrt{xy}=7 \\ x^2+y^2+xy=21 \end{cases}\right\}.$ *Ans.* $\begin{cases} x=4 \text{ or } 1, \\ y=1 \text{ or } 4. \end{cases}$

6. $\left.\begin{cases} x+y-\sqrt{xy}=6 \\ x^2+y^2+xy=84 \end{cases}\right\}.$ *Ans.* $\begin{cases} x=8 \text{ or } 2, \\ y=2 \text{ or } 8. \end{cases}$

7. $\left.\begin{cases} x+\sqrt{x^2-y^2}=8 \\ x-y=1 \end{cases}\right\}.$ *Ans.* $\begin{cases} x=13 \text{ or } 5, \\ y=12 \text{ or } 4. \end{cases}$

8. $\left.\begin{cases} x+y=10 \\ \sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{5}{2} \end{cases}\right\}.$ *Ans.* $\begin{cases} x=8 \text{ or } 2, \\ y=2 \text{ or } 8. \end{cases}$

9. $\left.\begin{cases} \sqrt{x}-\sqrt{y}=2\sqrt{xy} \\ x+y=20 \end{cases}\right\}.$

Ans. $\begin{cases} x=10\pm 4\sqrt{6} \text{ or } 10\mp\dfrac{5}{2}\sqrt{15}, \\ y=10\mp 4\sqrt{6} \text{ or } 10\pm\dfrac{5}{2}\sqrt{15}. \end{cases}$

10. $\left.\begin{cases} x+y=a\sqrt{xy} \\ x-y=c\sqrt{\dfrac{x}{y}} \end{cases}\right\}.$ *Ans.* $\begin{cases} x=\dfrac{a^2c-2c\pm ac\sqrt{a^2-4}}{\pm 2\sqrt{a^2-4}}, \\ y=\pm\dfrac{c}{\sqrt{a^2-4}}. \end{cases}$

11. $\left.\begin{cases} xy-(x+y)=54 \\ \left(\dfrac{3x}{x+y}\right)^{\frac{1}{2}}+\left(\dfrac{x+y}{3x}\right)^{\frac{1}{2}}=2 \end{cases}\right\}.$ *Ans.* $\begin{cases} x=6 \text{ or } -4\frac{1}{2}, \\ y=12 \text{ or } -9. \end{cases}$

12. $\left.\begin{cases} y(x+y)^{-\frac{3}{2}}+\dfrac{(x+y)^{\frac{1}{2}}}{y}=\dfrac{17}{4}(x+y)^{-\frac{1}{2}} \\ x=y^2+2 \end{cases}\right\}.$

Ans. $\begin{cases} x=6 \text{ or } 3, \\ y=2 \text{ or } 1. \end{cases}$

13. $\left\{\begin{array}{l} x^{\frac{1}{4}} + y^{\frac{1}{3}} = 5 \\ x^{\frac{1}{2}} + y^{\frac{2}{3}} = 13 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 81 \text{ or } 16, \\ y = 8 \text{ or } 27. \end{array}\right.$

14. $\left\{\begin{array}{l} x^3 + y^3 = 189 \\ x + y + \sqrt{x+y} = 12 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 5 \text{ or } 4, \\ y = 4 \text{ or } 5. \end{array}\right.$

15. $\left\{\begin{array}{r} \sqrt{y} - \sqrt{y-x} = \sqrt{a-x} \\ 2\sqrt{y-x} = 3\sqrt{a-x} \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = \frac{4}{5}a, \\ y = \frac{5}{4}a. \end{array}\right.$

16. $\left\{\begin{array}{l} x + y - \sqrt{\dfrac{x+y}{x-y}} = \dfrac{6}{x-y} \\ x^2 + y^2 = 41 \end{array}\right\}.$

Ans. $\left\{\begin{array}{l} x = \pm 5 \text{ or } \pm 3\sqrt{\frac{5}{2}}, \\ y = \pm 4 \text{ or } \pm \sqrt{\frac{37}{2}}. \end{array}\right.$

GROUPS WITH MORE THAN TWO UNKNOWN QUANTITIES.

394. *EXAMPLES.*

1 Solve the equations

$$x^2 + xy + y^2 = 37 \quad . \; . \; . \quad (1),$$

$$x^2 + xz + z^2 = 28 \quad . \; . \; . \quad (2),$$

$$y^2 + yz + z^2 = 19 \quad . \; . \; . \quad (3).$$

Subtracting (2) from (1),

$$(y - z)x + y^2 - z^2 = 9;$$

whence by factoring,

$$(y - z)(x + y + z) = 9 \quad . \; . \; . \quad (4).$$

Subtracting (3) from (2) and factoring the result,

$$(x - y)(x + y + z) = 9 \quad . \; . \; . \quad (5).$$

Combining (4) and (5),

$$y - z = x - y;$$

whence, $$x + z = 2y \quad . \quad . \quad . \quad (6).$$

Combining (5) and (6),

$$(x - y)\, 3y = 9;$$

whence, $$x = \frac{3}{y} + y \quad . \quad . \quad . \quad (7).$$

Combining (1) and (7),

$$\left(\frac{3}{y} + y\right)^2 + 3 + y^2 + y^2 = 37;$$

whence, $$y = \pm 3 \quad \text{or} \quad \pm \frac{1}{3}\sqrt{3}.$$

Substituting these values in (7),

$$x = \pm 4 \quad \text{or} \quad \pm \frac{10}{3}\sqrt{3}.$$

Substituting for x and y their values, we find from (6)

$$z = \pm 2 \quad \text{or} \quad \mp \frac{8}{3}\sqrt{3}.$$

2. Solve the equations

$$x + y + z = a \quad . \quad . \quad . \quad (1),$$

$$x^2 + y^2 + z^2 = b^2 \quad . \quad . \quad . \quad (2),$$

$$xy = cz \quad . \quad . \quad . \quad (3).$$

Transposing z in (1),

$$x + y = a - z;$$

whence, $$x^2 + 2xy + y^2 = a^2 - 2az + z^2 \quad . \quad . \quad . \quad (4).$$

Transposing z^2 in (2),

$$x^2 + y^2 = b^2 - z^2 \quad . \quad . \quad . \quad (5).$$

Subtracting (5) from (4),

$$2xy = a^2 + 2z^2 - 2az - b^2 \quad . \quad . \quad . \quad (6).$$

Combining (3) and (6),

$$2cz = a^2 + 2z^2 - 2az - b^2 \quad . \quad . \quad . \quad (7);$$

whence, $$z = \frac{a + c \pm \sqrt{(a+c)^2 - 2(a^2 - b^2)}}{2}.$$

Substituting these values for z in (1) and (3),

$$x + y + \frac{a + c \pm \sqrt{(a+c)^2 - 2(a^2 - b^2)}}{2} = a \quad . \quad . \quad . \quad (8),$$

$$xy = \frac{ac + c^2 \pm c\sqrt{(a+c)^2 - 2(a^2 - b^2)}}{2} \quad . \quad . \quad . \quad (9).$$

From (8) and (9) the values of x and y may be found.

Solve the following groups of equations:

3. $\left\{\begin{aligned} xy^2z^3 &= 4725 \\ \frac{yz^2}{x} &= \frac{45}{7} \\ \frac{z}{x^2y} &= \frac{3}{245} \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x &= 7, \\ y &= 5, \\ z &= 3. \end{aligned}\right.$

4. $\left\{\begin{aligned} x + y + z &= 13 \\ x^2 + y^2 + z^2 &= 61 \\ 2yz &= x(z + y) \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x &= 9 \text{ or } 4, \\ y &= 2 \pm \sqrt{-14} \text{ or } 3, \\ z &= 2 \mp \sqrt{-14} \text{ or } 6. \end{aligned}\right.$

5. $\left\{\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 \\ \frac{x}{a} + \frac{z}{c} &= 1 \\ \frac{y}{c} &= \frac{b}{z} \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x &= 0 \text{ or } 2a, \\ y &= b \text{ or } -b, \\ z &= c \text{ or } -c. \end{aligned}\right.$

6. $\left\{\begin{aligned} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 9 \\ \frac{2}{x} + \frac{3}{y} &= 13 \\ 8x + 3y &= 5 \end{aligned}\right\}.$ *Ans.* $\left\{\begin{aligned} x &= \frac{1}{2} \text{ or } \frac{5}{26}, \\ y &= \frac{1}{3} \text{ or } \frac{15}{13}, \\ z &= \frac{1}{4} \text{ or } \frac{15}{44}. \end{aligned}\right.$

7. $\left.\begin{cases} y + z = \dfrac{1}{x} \\ z + x = \dfrac{1}{y} \\ x + y = \dfrac{1}{z} \end{cases}\right\}.$ *Ans.* $x = y = z = \pm\sqrt{\dfrac{1}{2}}.$

8. $\left.\begin{cases} x^2 + xy + xz = 27 \\ xy + y^2 + yz = 18 \\ xz + yz + z^2 = 36 \end{cases}\right\}.$ *Ans.* $\begin{cases} x = \pm 3, \\ y = \pm 2, \\ z = \pm 4. \end{cases}$

9. $\left.\begin{cases} y = \dfrac{z}{x} \\ z = \dfrac{v}{y} \\ v = \dfrac{a}{x} \\ bx = vy \end{cases}\right\}.$ *Ans.* $\begin{cases} x = \dfrac{\pm\sqrt{a}}{\sqrt[3]{b}}, \\ y = \sqrt[3]{b}, \\ z = \pm\sqrt{a}, \\ v = \pm\sqrt{a}\sqrt[3]{b}. \end{cases}$

10. $\left.\begin{cases} xyz = 6 \\ xyv = 8 \\ xzv = 12 \\ yzv = 24 \end{cases}\right\}.$ *Ans.* $\begin{cases} x = 1, \\ y = 2, \\ z = 3, \\ v = 4. \end{cases}$

395. ***PROBLEMS.***

1. The sum of two numbers is equal to nine times their difference; and if the greater be subtracted from their product, the remainder will be equal to twelve times the quotient obtained by dividing the greater by the less: find the numbers.

Ans. 5 and 4.

2. The sum of two numbers is equal to a times their difference; and if the greater be subtracted from their product, the remainder will be equal to b times the quotient obtained by dividing the greater by the less: find the numbers.

Ans. $\dfrac{a+1}{a-1}\left(\dfrac{1}{2} \pm \sqrt{b + \dfrac{1}{4}}\right)$ and $\dfrac{1}{2} \pm \sqrt{b + \dfrac{1}{4}}.$

3. Find two numbers whose difference is equal to two-ninths of the greater, and the difference of whose squares is 128.

Ans. 18 and 14.

4. Find two numbers whose difference is equal to $\frac{1}{n}$ of the greater, and the difference of whose squares is a.

$$Ans.\ \pm n\sqrt{\frac{a}{2n-1}} \text{ and } \pm(n-1)\sqrt{\frac{a}{2n-1}}.$$

5. The sum of two numbers is 16, and the quotient obtained by dividing the greater by the less is $2\frac{7}{9}$ times the quotient obtained by dividing the less by the greater: find the numbers.

Ans. 10 and 6.

6. The sum of two numbers is a, and the quotient obtained by dividing the greater by the less is b times the quotient obtained by dividing the less by the greater: find the numbers.

$$Ans.\ \frac{a(b \pm \sqrt{b})}{b-1} \text{ and } \frac{-a(1 \pm \sqrt{b})}{b-1}.$$

7. The difference of two numbers is 15, and half their product is equal to the cube of the smaller number: find the numbers.

Ans. 18 and 3.

8. The difference of two numbers is d, and half their product is equal to the cube of the smaller number: find the numbers.

$$Ans.\ \frac{1+4d \pm \sqrt{8d+1}}{4} \text{ and } \frac{1 \pm \sqrt{8d+1}}{4}.$$

9. The product of two numbers is 24, and the product of their sum and their difference is 20: find the numbers.

Ans. 6 and 4.

10. The product of two numbers is a, and the product of their sum and their difference is b: find the numbers.

$$Ans.\ \pm\sqrt{\frac{b \pm \sqrt{4a^2+b^2}}{2}} \text{ and } \pm\sqrt{\frac{-b \pm \sqrt{4a^2+b^2}}{2}}.$$

11. The product of two numbers is 18 times their difference, and the sum of their squares is 117: find the numbers.

Ans. 9 and 6.

12. The product of two numbers is m times their difference, and the sum of their squares is a: find the numbers.

Ans. $c \pm \sqrt{c(2m + c)}$ and $-c \pm \sqrt{c(2m + c)}$,

where $c = \dfrac{-m \pm \sqrt{a + m^2}}{2}$.

13. Two persons, A and B, bought a farm containing 600 acres, for which they paid \$600, each paying \$300. A paid 75 cents more per acre than B in order to be permitted to take his share from the best land. How many acres did each get, and at what price per acre? *Ans.* A 200 acres at \$1.50, B 400 acres at \$0.75.

14. A certain number of workmen in 8 hours carried a pile of stones from one place to another. Had there been 8 more workmen, and had each carried each time 5 pounds less, the pile would have been removed in 7 hours; but if there had been 8 workmen less, and had each carried each time 11 pounds more, it would have required 9 hours to remove the pile. How many workmen were employed, and how many pounds did each carry at a time?

Ans. 28 workmen, and each carried 45 pounds; or 36 workmen, and each carried 77 pounds.

15. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel be increased one yard, the fore-wheel will make only 4 revolutions more than the hind-wheel in going the same distance. What is the circumference of each?

Ans. Fore-wheel, 4 yds.; hind-wheel, 5 yds.

16. What number is that, which being divided by the product of its two digits gives 2 for the quotient, and if 27 be added to it the order of the digits will be inverted? *Ans.* 36.

17. A sets off from London to York, and B at the same time from York to London, each traveling at a uniform rate. A reaches York 16 hours, and B reaches London 36 hours after they have met on the road. Find in what time each has performed the journey. *Ans.* A in 40 hours, B in 60 hours.

18. A man had \$1300, which he divided into two parts, and placed at interest at such rates that the incomes from them were equal. If he had put out the first part at the same rate as the second, he would have drawn for this part \$36 interest; and if he had put out the second part at the same rate as the first, he would have drawn for it \$49 interest. Find the rates of interest.

Ans. 6 per cent. for the larger part, and 7 per cent. for the smaller.

19. A and B engage to reap a field for \$24; and as A alone could reap it in 9 days they promise to complete it in 5 days. They found, however, that they were obliged to call in C to assist them 2 days in order to complete the work in the stipulated time, in consequence of which B received \$1 less than he would have done if he and A, without the assistance of C, had continued until they completed the work. In what time could B or C alone reap the field? *Ans.* B in 15 days, C in 18 days.

20. A number consists of three digits. The sum of the squares of the digits is 104; the square of the middle digit exceeds twice the product of the other two by 4; and if 594 be subtracted from the number the order of the digits will be inverted: find the number. *Ans.* 862.

396. SYNOPSIS FOR REVIEW.

CHAPTER XV. SIMULTANEOUS EQUATIONS.

- GROUPS AND PAIRS.
- DEGREE OF AN EQUATION.
- GENERAL FORM OF AN EQUATION OF THE SECOND DEGREE INVOLVING TWO UNKNOWN QUANTITIES.
- PAIRS CONSISTING OF ONE EQUATION OF THE FIRST DEGREE AND ONE OF THE SECOND DEGREE.
- PARTICULAR SYSTEMS.
- PAIRS OF EQUATIONS INVOLVING RADICALS.
- GROUPS INVOLVING MORE THAN TWO UNKNOWN QUANTITIES AND ONE OR MORE EQUATIONS OF A HIGHER DEGREE THAN THE FIRST.

CHAPTER XVI.

RATIO, PROPORTION, AND VARIATION.

RATIO.

397. *The Ratio* of two quantities is the quotient arising from dividing the first by the second. Thus, the ratio of 6 to 3 is 2, and the ratio of a to b is $\frac{a}{b}$.*

398. *The Sign of ratio* is the colon. Thus, $a:b$ is read *the ratio of a to b.*

399. *The Terms of a ratio* are the quantities compared. Thus, in the expression $a:b$, the terms of the ratio are a and b.

400. *The Antecedent of a ratio* is its first term, and the ***Consequent*** of a ratio is its second term.

The antecedent and consequent of a ratio together form a ***Couplet.***

401. *A Simple Ratio* is one whose terms are entire. Thus, $a:b$ is a simple ratio.

* Ratio as thus defined is sometimes called *geometrical ratio* or *ratio by quotient*, to distinguish it from *arithmetical ratio* or *ratio by difference.* The *arithmetical* ratio of a to b is $a-b$. The sign of arithmetical ratio is $\cdot\cdot$. Thus, $a \cdot\cdot b$ is read *the arithmetical ratio of a to b.*

When the word *ratio* is used without modification it is understood to mean *geometrical ratio.*

402. *A Complex Ratio* is one in which at least one of the terms involves a fraction. Thus, $\frac{a}{c}:b$ and $\frac{a}{c}:\frac{b}{a}$ are complex ratios.

403. *A Compound Ratio* is the ratio of the products of the corresponding terms of two or more ratios. Thus, the ratio *compounded* of $a:b$ and $c:d$ is $ac:bd$.

A compound ratio does not differ in its *nature* from any other ratio. The term is used to denote the origin of the ratio.

404. *The Duplicate Ratio* of two quantities is the ratio of their squares. Thus, $a^2:b^2$ is the duplicate ratio of a to b.

405. *The Triplicate Ratio* of two quantities is the ratio of their cubes. Thus, $a^3:b^3$ is the triplicate ratio of a to b.

406. *The Subduplicate Ratio* of two quantities is the ratio of their square roots. Thus, $\sqrt{a}:\sqrt{b}$ is the subduplicate ratio of a to b.

407. *The Subtriplicate Ratio* of two quantities is the ratio of their cube roots. Thus, $\sqrt[3]{a}:\sqrt[3]{b}$ is the subtriplicate ratio of a to b.

408. *The Direct Ratio* of two quantities is the quotient arising from dividing the antecedent by the consequent. Thus, the direct ratio of a to b is $\frac{a}{b}$.

409. *The Inverse* or ***Reciprocal Ratio*** of two quantities is the *direct* ratio of their *reciprocals,* or the quotient arising from dividing the consequent by the antecedent. Thus, the inverse ratio of a to b is $\frac{1}{a}:\frac{1}{b}$ or $\frac{b}{a}$.

410. A ratio is called a ratio of ***Greater Inequality,*** of ***Less Inequality,*** or of ***Equality,*** according as the antecedent is *greater* than, *less* than, or *equal to,* the consequent.

411. *EXAMPLES.*

1. Find the ratio of $a^2 - b^2$ to $a + b$.

$$a^2 - b^2 : a + b = \frac{a^2 - b^2}{a + b} = a - b.$$

2. Find the inverse ratio of $a^2 - b^2$ to $a + b$. *Ans.* $\frac{1}{a - b}$.

3. Find the ratio which is compounded of $3:5$ and $7:9$.
Ans. $\frac{7}{15}$.

4. Find the subduplicate ratio of 100 to 144. *Ans.* $\frac{5}{6}$.

5. Show that $a:b$ is the duplicate of $a + c : b + c$ if $c^2 = ab$.

PROPORTION.

412. ***A Proportion*** is an equation in which each member is a ratio, both terms of which are expressed.*

The equality of the two ratios may be indicated either by the sign $=$, or by the double colon $::$. Thus, we may indicate that the ratio of 8 to 4 is equal to that of 6 to 3 in any of the following ways:

$$\left\{\begin{array}{l} 8:4 = 6:3 \\ 8:4::6:3 \\ \frac{8}{4} = \frac{6}{3} \\ 8 \div 4 = 6 \div 3 \end{array}\right\}.$$

This proportion, in any of its forms, is read *the ratio of 8 to 4 is equal to the ratio of 6 to 3*, or, *8 is to 4 as 6 is to 3*.

* *A geometrical proportion* is one in which the ratios are geometrical.

An arithmetical proportion is one in which the ratios are arithmetical. Thus, $6 \cdot\cdot 5 = 9 \cdot\cdot 8$ is an arithmetical proportion.

When the word *proportion* is used without modification, it is understood to mean *geometrical proportion*.

413. ***The Terms*** of a proportion are the four quantities which are compared. The first and second terms form the first *couplet*; the third and fourth, the second *couplet.*

414. ***The Antecedents*** in a proportion are the *first* and *third* terms.

415. ***The Consequents*** in a proportion are the *second* and *fourth* terms.

416. ***The Extremes*** in a proportion are the *first* and *fourth* terms.

417. ***The Means*** in a proportion are the *second* and *third* terms.

418. If four quantities a, b, c, and d are so related that

$$a:b=c:d,$$

d is said to be a ***Fourth Proportional*** to a, b, and c.

419. If three quantities a, b, and c are so related that

$$a:b=b:c,$$

c is said to be a ***Third Proportional*** to a and b.

420. If three quantities a, b, and c are so related that

$$a:b=b:c,$$

b is said to be a ***Mean Proportional*** between a and c.

421. ***A Continued Proportion*** is a continued equation in which each member is a ratio, both terms of which are expressed. Thus, $a:b=c:d=e:f=g:h$ is a continued proportion.

422. If four quantities a, b, c, and d are so related that

$$a:b=\frac{1}{c}:\frac{1}{d},$$

they are said to be ***Inversely or Reciprocally Proportional.***

423. ***Equimultiples*** of two or more quantities are the products obtained by multiplying each of them by the same quantity. Thus, ma and mb are equimultiples of a and b.

424. A proportion is taken by ***Alternation*** when the means or the extremes are made to exchange places. Thus, if $a:b=c:d$, we have by alternation, either $a:c=b:d$, or $d:b=c:a$.

425. A proportion is taken by ***Inversion*** when the terms of each couplet are made to exchange places. Thus, if $a:b=c:d$, we have by inversion, $b:a=d:c$.

426. A proportion is taken by ***Composition*** when the sum of the terms of each couplet is compared with either term of that couplet, the same order being observed in the two couplets. Thus, if $a:b=c:d$, we have by composition, either $a+b:a=c+d:c$, or $a+b:b=c+d:d$.

427. A proportion is taken by ***Division*** when the difference of the terms of each couplet is compared with either term of that couplet, the same order being observed in the two couplets. Thus, if $a:b=c:d$, we have by division, either $a-b:a=c-d:c$, or $a-b:b=c-d:d$.

428. *In every proportion the product of the extremes is equal to the product of the means.*

Let $$a:b=c:d;$$

then $$\frac{a}{b}=\frac{c}{d}\ (\mathbf{412});$$

whence, $$ad=bc.$$

COR.—If the means are equal, as in the proportion $a:b=b:c$, we have $b^2=ac$, whence $b=\sqrt{ac}$; that is, *a mean proportional between two quantities is equal to the square root of their product.*

429. *If the product of two quantities is equal to the product of two others, either two may be made the extremes, and the other two the means, of a proportion.*

Let $$ad = bc \quad . \quad . \quad . \quad (1);$$

then, dividing both members by cd,

$$\frac{a}{c} = \frac{b}{d}, \quad \text{or} \quad a:c = b:d \quad . \quad . \quad . \quad (2).$$

In like manner it may be shown that

$$a:b = c:d \quad . \quad . \quad . \quad (3),$$
$$c:a = d:b \quad . \quad . \quad . \quad (4),$$
$$c:d = a:b \quad . \quad . \quad . \quad (5),$$
$$d:b = c:a \quad . \quad . \quad . \quad (6),$$

and so on.

Cor. 1.—Any one of these proportions may be inferred from any other. Thus, from (2),

$$ad = bc,$$

from which any one of the proportions may be derived.

Cor. 2.—Since (3) may be derived from (2), it follows that

If four quantities are in proportion, they will be in proportion by alternation.

Cor. 3.—Since (4) may be derived from (2), it follows that

If four quantities are in proportion, they will be in proportion by inversion.

430. *Equimultiples of two quantities are in the same ratio as the quantities themselves.*

$$\frac{ma}{mb} = \frac{a}{b};$$

$\therefore$ $$ma:mb = a:b.$$

Cor.—If $$m = 1 \pm \frac{p}{q},$$

we have $$a \pm \frac{p}{q}a : b \pm \frac{p}{q}b = a:b;$$

that is,

If two quantities be increased or diminished by like parts of each, the results will be in the same ratio as the quantities themselves.

431. *Any equimultiples of one couplet of a proportion are in the same ratio as any equimultiples of the other couplet.*

Let $$a : b = c : d;$$

then $$\frac{a}{b} = \frac{c}{d};$$

whence, $$\frac{ma}{mb} = \frac{nc}{nd};$$

$\therefore$ $$ma : mb = nc : nd.$$

COR.—If $m = 1 \pm \frac{p}{q}$ and $n = 1 \pm \frac{p'}{q'}$, we have

$$a \pm \frac{p}{q}a : b \pm \frac{p}{q}b = c \pm \frac{p'}{q'}c : d \pm \frac{p'}{q'}d;$$

that is,

If the terms of the first couplet of a proportion be increased or diminished by like parts of each, and if the terms of the second couplet be increased or diminished by any other like parts of each, the results will be in proportion.

432. *Any equimultiples of the antecedents of a proportion are in the same ratio as any equimultiples of the consequents.*

Let $$a : b = c : d \quad . \quad . \quad . \quad (1);$$

then $$\frac{a}{b} = \frac{c}{d} \quad . \quad . \quad . \quad (2).$$

Multiplying (2) by $\frac{m}{n}$,

$$\frac{ma}{nb} = \frac{mc}{nd};$$

whence, $$ma : nb = mc : nd,$$

which by alternation becomes

$$ma : mc = nb : nd.$$

433. Axiom.—*Ratios that are equal to the same or equal ratios are equal to each other.*

Thus, if $a : b = c : d$,

$e : f = g : h$,

and $a : b = e : f$;

then $c : d = g : h$.

434. *If the ratio of the antecedents of one proportion is equal to the ratio of the antecedents of another proportion, the ratio of the consequents of the one will be equal to the ratio of the consequents of the other, and conversely.*

Let $a : b = c : d \quad . \quad . \quad . \quad (1)$,

$e : f = g : h \quad . \quad . \quad . \quad (2)$,

and $a : c = e : g \quad . \quad . \quad . \quad (3)$;

then will $b : d = f : h$.

For, from (1), $a : c = b : d$ (**429**, Cor. 2),

and from (2), $e : g = f : h$;

$\therefore$ $b : d = f : h$ (**433**).

In like manner it may be shown that, if

$a : b = c : d$,

$e : f = g : h$,

and $b : d = f : h$;

then will $a : c = e : g$.

435. *If four quantities are in proportion, they will be in proportion by composition or division.*

Let $a : b = c : d \quad . \quad . \quad . \quad (1)$,

then $\frac{a}{b} = \frac{c}{d} \quad . \quad . \quad . \quad (2)$;

whence, $\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$,

or, $$\frac{a \pm b}{b} = \frac{c \pm d}{d} \quad . \quad . \quad . \quad (3);$$

whence, $$a \pm b : b = c \pm d : d \quad . \quad . \quad . \quad (4).$$

Dividing (3) by (2),

$$\frac{a \pm b}{a} = \frac{c \pm d}{c} \quad . \quad . \quad . \quad (5);$$

whence, $$a \pm b : a = c \pm d : c \quad . \quad . \quad . \quad (6).$$

Cor.—Separating (6),

$$a + b : a = c + d : c,$$

and $$a - b : a = c - d : c;$$

$$\therefore \quad a + b : a - b = c + d : c - d \text{ } (\mathbf{434});$$

that is,

If four quantities are in proportion, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

436. *The sum of any number of the antecedents of a continued proportion is to the sum of the corresponding consequents as any antecedent is to its consequent.*

Let $$a : b = c : d = e : f = g : h = \&c.,$$

and let r denote the ratio; then

$$r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = \&c.;$$

whence, $\quad a = br, \quad c = dr, \quad e = fr, \quad g = hr, \quad \&c.;$

whence, by addition,

$$a + c + e + g + \&c. = (b + d + f + h + \&c.)\, r;$$

whence, $$\frac{a + c + e + g + \&c.}{b + d + f + h + \&c.} = r = \frac{a}{b} = \frac{c}{d} = \&c.;$$

$$\therefore \quad a + c + e + g + \&c. : b + d + f + h + \&c. = a : b = c : d = \&c.$$

437. *The products of the corresponding terms of two or more proportions are in proportion.*

Let $$a : b = c : d,$$

$$e : f = g : h,$$

and $$m : n = p : q;$$

then $$\frac{a}{b} = \frac{c}{d},$$

$$\frac{e}{f} = \frac{g}{h},$$

and $$\frac{m}{n} = \frac{p}{q};$$

whence, by multiplication,

$$\frac{aem}{bfn} = \frac{cgp}{dhq};$$

$$\therefore \quad aem : bfn = cgp : dhq.$$

438. *The quotients of the corresponding terms of two proportions are in proportion.*

Let $$a : b = c : d,$$

and $$e : f = g : h;$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{e}{f} = \frac{g}{h};$$

whence, $$\frac{af}{be} = \frac{ch}{dg};$$

or, $$\frac{a}{e} \times \frac{d}{h} = \frac{c}{g} \times \frac{b}{f};$$

$$\therefore \quad \frac{a}{e} : \frac{b}{f} = \frac{c}{g} : \frac{d}{h}.$$

439. *Like powers or like roots of the terms of a proportion are in proportion.*

Let $$a:b=c:d \quad . \quad . \quad . \quad (1),$$

then $$\frac{a}{b}=\frac{c}{d} \quad . \quad . \quad . \quad (2).$$

Raising (2) to the n^{th} power,

$$\frac{a^n}{b^n}=\frac{c^n}{d^n};$$

$$\therefore \quad a^n:b^n=c^n:d^n.$$

Extracting the n^{th} root of (2),

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{c}}{\sqrt[n]{d}};$$

$$\therefore \quad \sqrt[n]{a}:\sqrt[n]{b}=\sqrt[n]{c}:\sqrt[n]{d}.$$

440. *PROBLEMS.*

1. Find two numbers, the greater of which shall be to the less as their sum is to 21, and as their difference is to 3.

Let $x=$ the greater number, and $y=$ the less; then by the conditions of the problem,

$$\left\{\begin{array}{ll} x:y=x+y:21 \quad . \quad . \quad . & (1) \\ x:y=x-y:\ \ 3 \quad . \quad . \quad . & (2) \end{array}\right\};$$

whence, $$\left\{\begin{array}{ll} 21x=xy+y^2 \quad . \quad . \quad . & (3) \\ 3x=xy-y^2 \quad . \quad . \quad . & (4) \end{array}\right\}.$$

Adding (3) and (4),

$$24x=2xy \quad . \quad . \quad . \quad (5);$$

whence, $$y=12.$$

Substituting 12 for y in (4), we find

$$x=16.$$

2. Divide the number 14 into two such parts that the quotient arising from dividing the greater by the less shall be to the quotient arising from dividing the less by the greater as 16 is to 9.

Let $x =$ the greater number, then will $14 - x =$ the less.

By the conditions of the problem,

$$\frac{x}{14-x} : \frac{14-x}{x} = 16 : 9 \quad . \quad . \quad . \quad (1).$$

Multiplying the terms of the first couplet by $(14 - x)\,x$,

$$x^2 : (14 - x)^2 = 16 : 9 \quad . \quad . \quad . \quad (2) \quad \mathbf{(430)};$$

whence, $$x : 14 - x = 4 : 3 \quad . \quad . \quad . \quad (3) \quad \mathbf{(439)}.$$

From this proportion we find

$$x = 8 \quad \text{and} \quad 14 - x = 6.$$

3. The product of two numbers is 112, and the difference of their cubes is to the cube of their difference as 31 is to 3. What are the numbers?

Let $x =$ the greater number, and $y =$ the less; then by the conditions of the problem,

$$\left\{\begin{array}{rl} xy = 112 & . \quad . \quad . \quad (1) \\ x^3 - y^3 : (x - y)^3 = 31 : 3 & . \quad . \quad . \quad (2) \end{array}\right\}.$$

From (2),

$$x^2 + xy + y^2 : x^2 - 2xy + y^2 = 31 : 3 \quad . \quad . \quad . \quad (3) \quad \mathbf{(430)};$$

whence, $$3xy : (x - y)^2 = 28 : 3 \quad . \quad . \quad . \quad (4).$$

Combining (1) and (4),

$$336 : (x - y)^2 = 28 : 3 \quad . \quad . \quad . \quad (5);$$

whence, $$x - y = 6 \quad . \quad . \quad . \quad (6).$$

Combining (1) and (6), we find

$$x = 14, \quad y = 8.$$

4. What two numbers are those whose difference is to their sum as 2 is to 9, and whose sum is to their product as 18 is to 77?

Let $x =$ the greater number, and $y =$ the less; then by the conditions of the problem,

$$\left\{\begin{array}{ll} x - y : x + y = \ \ 2 : \ 9 \quad . \ . \ . & (1) \\ x + y : xy \qquad = 18 : 77 \quad . \ . \ . & (2) \end{array}\right\}.$$

From (1),

$$2x : 2y = 11 : 7 \quad . \ . \ . \quad (3) \ (\mathbf{435}, \text{Cor.});$$

whence,
$$y = \frac{7x}{11} \quad . \ . \ . \quad (4).$$

Combining (2) and (4),

$$\frac{18x}{11} : \frac{7x^2}{11} = 18 : 77 \quad . \ . \ . \quad (5);$$

whence,
$$x = 11.$$

Substituting in (4), we find

$$y = 7.$$

5. Two numbers have such a relation that if 4 be added to each, the results will be in the ratio of 3 to 4; and if 4 be subtracted from each, the results will be in the ratio of 1 to 4. What are the numbers? *Ans.* 5 and 8.

6. Divide the number 27 into two such parts that their product shall be to the sum of their squares as 20 is to 41.

Ans. 12 and 15.

7. Two numbers are in the ratio of 3 to 2. If 6 be added to the greater and subtracted from the less, the results will be in the ratio of 3 to 1. What are the numbers? *Ans.* 24 and 16.

8. The number 20 is divided into two parts which are to each other in the duplicate ratio of 3 to 1. Find the mean proportional between these parts. *Ans.* 6.

9. The sum of the cubes of two numbers is to the difference of their cubes as 559 is to 127, and the product obtained by multiplying the less by the square of the greater is equal to 294. What are the numbers? *Ans.* 7 and 6.

10. The cube of the first of two numbers is to the square of the second as 3 is to 1, and the cube of the second is to the square of the first as 96 is to 1. What are the numbers?

Ans. 12 and 24.

11. Given $m + x : n + x = p + x : q + x$ to find x.

Ans. $\dfrac{np - mq}{m + q - n - p}$.

VARIATION.

441. One quantity varies *directly* as another when the two quantities have such a relation that one increases or decreases in the same ratio as the other. Thus, in uniform motion, the space described varies directly as the time.

Sometimes, for brevity, we omit the word *directly*, and say simply that one quantity varies as another.

442. ***The Sign*** of variation is the symbol $\propto$. Thus, the expression $s \propto t$ is read *s varies as t*.

443. ***A Variation*** consists of two expressions connected by the sign $\propto$.

When a variation immediately follows the word *let*, the sign $\propto$ is equivalent to the word *vary*.

444. One quantity varies *inversely* as another when the first varies as the *reciprocal* of the second. Thus, in uniform motion, if the space (s) is constant, the time (t) varies inversely as the velocity (v); that is, $t \propto \frac{1}{v}$.

445. One quantity varies as two others *jointly* when the first varies as the product of the others. Thus, in uniform motion, the space (s) varies as the time (t) and velocity (v) jointly; that is, $s \propto vt$.

446. One quantity varies *directly* as a second quantity and *inversely* as a third, when the first varies as the quotient obtained by dividing the second by the third. Thus, in uniform motion,

the time (t) varies directly as the space (s) and inversely as the velocity (v); that is, $t \propto \frac{s}{v}$.

447. *If one quantity varies as another, either of them is equal to the product obtained by multiplying the other by some constant quantity.*

Let $$x \propto y,$$

and suppose $y = b$ when $x = a$; then

$$x : a = y : b \text{ } (\mathbf{441});$$

whence, $$x = \frac{a}{b} y = my,$$

where m is equal to the constant $\frac{a}{b}$.

448. *If one variable quantity is equal to the product obtained by multiplying another by a constant, the first varies as the second.*

Let $$x = my,$$

and suppose $y = b$ when $x = a$; then

$$a = mb;$$

$\therefore$ $$x : a = y : b;$$

that is, $$x \propto y.$$

449. *If one quantity varies as a second, and the second as a third, the first varies as the third.*

Let $$\left\{ \begin{matrix} x \propto y \\ y \propto z \end{matrix} \right\};$$

then $$\left\{ \begin{matrix} x = my \\ y = nz \end{matrix} \right\} \text{ } (\mathbf{447});$$

whence, $$x = mnz;$$

$\therefore$ $$x \propto z \quad (\mathbf{448}).$$

450. *If each of two quantities varies as a third, their sum, their difference, or their mean proportional varies as the third.*

Let $$\left\{\begin{array}{l} x \propto z \\ y \propto z \end{array}\right\};$$

then $$\left\{\begin{array}{l} x = mz \\ y = nz \end{array}\right\};$$

whence,

$$x + y = (m + n)\,z, \quad x - y = (m - n)\,z, \quad \text{and} \quad \sqrt{xy} = z\sqrt{mn};$$

$$\therefore \qquad x + y \propto z, \qquad x - y \propto z, \qquad \text{and} \qquad \sqrt{xy} \propto z.$$

451. *If one quantity varies as two others jointly, either of the latter varies directly as the first and inversely as the other.*

Let $$x \propto yz;$$

then $$x = myz;$$

whence, $$y = \frac{1}{m}\cdot\frac{x}{z}, \text{ and } z = \frac{1}{m}\cdot\frac{x}{y};$$

$$\therefore \qquad y \propto \frac{x}{z}, \qquad \text{and} \quad z \propto \frac{x}{y}.$$

452. *If one quantity varies as another, either varies as any multiple of the other.*

Let $$x \propto y;$$

then $$x = my = \frac{m}{n}\cdot ny;$$

$$\therefore \qquad x \propto ny.$$

Again, since $x = my$,

$$y = \frac{1}{m}\cdot x = \frac{1}{mn}\cdot nx;$$

$$\therefore \qquad y \propto nx.$$

453. *If both members of a variation be multiplied or divided by the same quantity, the results will vary as each other.*

Let $$x \propto y \quad . \quad . \quad . \quad (1);$$

then $$x = my \quad . \quad . \quad . \quad (2).$$

Multiplying (2) by z,

$$xz = mzy;$$

$$\therefore \quad xz \propto zy.$$

Dividing (2) by z,

$$\frac{x}{z} = m\frac{y}{z};$$

$$\therefore \quad \frac{x}{z} \propto \frac{y}{z}.$$

COR.—Hence a variation may be cleared of fractions in the same way as an equation.

454. *The product of the first members of two or more variations varies as the product of their second members.*

Let
$$\left\{\begin{array}{l} x \propto y \\ t \propto z \\ w \propto v \end{array}\right\};$$

then
$$\left\{\begin{array}{l} x = my \\ t = nz \\ w = av \end{array}\right\};$$

whence,
$$xtw = amnvyz;$$

$$\therefore \quad xtw \propto vyz.$$

455. *The quotient of the first members of two variations varies as the quotient of their second members.*

Let
$$\left\{\begin{array}{l} x \propto y \\ z \propto t \end{array}\right\};$$

then
$$\left\{\begin{array}{l} x = my \\ z = nt \end{array}\right\};$$

whence,
$$\frac{x}{z} = \frac{m}{n} \cdot \frac{y}{t};$$

$$\therefore \quad \frac{x}{z} \propto \frac{y}{t}.$$

456. *Like powers or like roots of the members of a variation vary as each other.*

Let $x \propto y;$

then $x = my;$

whence, $x^n = m^n y^n,$ and $\sqrt[n]{x} = \sqrt[n]{m}\sqrt[n]{y};$

$\therefore$ $x^n \propto y^n,$ and $\sqrt[n]{x} \propto \sqrt[n]{y}.$

457. *PROBLEMS.*

1. If $y \propto x$ and $y = 3$ when $x = 1$, what is the value of y when $x = 3$? *Ans.* 9.

2. If $x \propto y$ and $x = 15$ when $y = 3$, what is the value of y in terms of x? *Ans.* $y = \frac{x}{5}$.

3. If $z \propto xy$ and $z = 1$ when $x = y = 1$, what is the value of z when $x = y = 2$? *Ans.* 4.

4. If $z \propto px + y$, and if $z = 3$ when $x = 1$ and $y = 2$, and $z = 5$ when $x = 2$ and $y = 3$, what is the value of p? *Ans.* 1.

5. If $x^2 \propto y^3$ and $x = 2$ when $y = 3$, what is the value of y in terms of x? *Ans.* $y = 3\sqrt[3]{\frac{x^2}{4}}$.

6. If $\left\{\begin{array}{l} y \propto t + v \\ t \propto x \\ v \propto \frac{1}{x} \end{array}\right\}$, and, if $y = 4$ when $x = 1$, and $y = 5$ when $x = 2$, what is the value of y in terms of x? *Ans.* $y = 2x + \frac{2}{x}$.

7. If $x \propto y$ when z is constant, and if $x \propto z$ when y is constant, how does x vary when both y and z are variable? *Ans.* $x \propto yz$.

458. SYNOPSIS FOR REVIEW.

CHAPTER XVI.

- RATIO . . . . .
  - Terms.—Antecedent.—Consequent.—Couplet.
  - Simple.—Complex.—Compound.
  - Duplicate.—Triplicate.—Sub-dup.—Sub-trip.
  - Direct.—Inverse or Reciprocal.—Of greater inequal.—Of less inequal.—Of equality.
- PROPORTION.
  - Geometrical. — Arithmetical. — How indicated? How read?
  - Terms. — Antecedent. — Consequent. — Extremes.—Means.
  - Fourth proportional.—Third proportional.—Mean proportional.
  - Continued proportion.
  - Inverse or reciprocal proportion.
  - Equimultiples. — Alternation.— Inversion.—Composition.—Division.
  - Propositions.
    - 428. Cor.
    - 429. Cor. 1, 2, 3.
    - 430. Cor.
    - 431. Cor.
    - 432.
    - 433.
    - 434.
    - 435. Cor.
    - 436.
    - 437.
    - 438.
    - 439.
- VARIATION.
  - Direct variation.
  - Sign of variation.
  - A variation.
  - Inverse variation.
  - Joint variation.
  - Direct and inverse variation combined.
  - Propositions.
    - 447.
    - 448.
    - 449.
    - 450.
    - 451.
    - 452.
    - 453. Cor.
    - 454.
    - 455.
    - 456.

CHAPTER XVII.

MATHEMATICAL INDUCTION.

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459. ***Mathematical Induction*** or ***Demonstrative Induction*** may be thus described: We prove that if a theorem is true in one case, whatever that case may be, it is true in another case which we may call the *next* case; we prove by trial that the theorem *is* true in a certain case; hence it is true in the next case, and hence in the next to that, and so on; hence it must be true in every case after that with which we began. This method of reasoning is exemplified in the demonstration of the following theorems:

460. *The sum of n consecutive integers beginning with* 1 *is* $\dfrac{n(n+1)}{2}$.

We see that this theorem is true in some cases; for example, $1+2=\dfrac{2(2+1)}{2}$, $1+2+3=\dfrac{3(3+1)}{2}$; we wish, however, to show that the theorem is true universally.

Suppose the theorem were known to be true for a certain value of n; that is, suppose for this value of n that

$$1+2+3+4+\ldots\ldots+n=\frac{n(n+1)}{2} \quad . \quad . \quad . \quad (1).$$

Adding $n+1$ to both members of (1),

$$1+2+3+4+\ldots+n+n+1=\frac{n(n+1)}{2}+n+1=$$

$$(n+1)\left[\frac{(n+1)+1}{2}\right] \quad . \quad . \quad . \quad (2).$$

Therefore, if the sum of n consecutive integers beginning with 1 is $\frac{n(n+1)}{2}$, the sum of $n+1$ such members will be $(n+1)\left[\frac{(n+1)+1}{2}\right]$. In other words, if the theorem is true when n is a certain number, whatever that number may be, it is true when we increase that number by 1. But we have seen by trial that the theorem is true when $n=3$; it is therefore true when $n=4$; it is therefore true when $n=5$; and so on. Hence the theorem must be universally true.

461. *The difference between the like powers of any two quantities is divisible by the difference between the quantities.*

Let a and b denote any two quantities, and let n be any positive integer; then will $a^n - b^n$ be divisible by $a-b$.

$$\frac{a^n - b^n}{a-b} = a^{n-1} + b\,\frac{(a^{n-1} - b^{n-1})}{a-b} \quad . \quad . \quad . \quad (1);$$

hence $a^n - b^n$ is divisible by $a-b$, if $a^{n-1} - b^{n-1}$ is divisible by it. Now $a-b$ is divisible by $a-b$; therefore $a^2 - b^2$ is divisible by $a-b$; therefore, again, $a^3 - b^3$ is divisible by $a-b$, and so on; hence $a^n - b^n$ is divisible by $a-b$, if n is a positive integer.

Performing the division indicated in (1), we obtain

$$\frac{a^n - b^n}{a-b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots\ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1} \ldots (2).$$

COR. 1.—The number of terms in the quotient is n.

COR. 2.—If $b=a$, each term of the quotient becomes equal to a^{n-1}; hence,

$$\left(\frac{a^n - b^n}{a-b}\right)_{b=a} = na^{n-1} \quad . \quad . \quad . \quad (3).$$

COR. 3.—Substituting c^2 for a and d^2 for b in (2), we have

$$\frac{c^{2n} - d^{2n}}{c^2 - d^2} = c^{2n-2} + c^{2n-4}d^2 + \ldots\ldots + c^2d^{2n-4} + d^{2n-2} \quad . \quad . \quad , (4);$$

hence,

The difference between the like even powers of any two quantities is divisible by the difference between the squares of the quantities.

COR. 4.—Multiplying both members of (4) by $c-d$, we obtain

$$\frac{c^{2n}-d^{2n}}{c+d}=(c-d)\,(c^{2n-2}+c^{2n-4}d^2+\ldots\ldots+c^2d^{2n-4}+d^{2n-2});$$

hence,

The difference between the like even powers of any two quantities is divisible by the sum of the quantities.

COR. 5.—Substituting c^m for a and d^m for b in (2), we have

$$\frac{c^{mn}-d^{mn}}{c^m-d^m}=c^{mn-m}+c^{mn-2m}d^m+\ldots\ldots+d^{mn-m};$$

hence,

The difference between the like powers of any two quantities is divisible by the difference between any other like powers of the two quantities, if the exponent in the first set of powers is divisible by that in the second set.

COR. 6.—When n is odd, we have

$$\frac{a^n-(-b)^n}{a-(-b)}=\frac{a^n+b^n}{a+b}.$$

Now by the theorem, $a^n-(-b)^n$ is divisible by $a-(-b)$; hence a^n+b^n is divisible by $a+b$ when n is odd; that is,

The sum of the like odd powers of any two quantities is divisible by the sum of the quantities.

Performing the division indicated, we obtain

$$\frac{a^n+b^n}{a+b}=a^{n-1}-a^{n-2}b+a^{n-3}b^2+\ldots\ldots-ab^{n-2}+b^{n-1}.$$

462. SYNOPSIS FOR REVIEW.

CHAPTER XVII. MATHEMATICAL INDUCTION. THEOREMS. **460.** **461.** *Cor.* 1, 2, 3, 4, 5, 6.

CHAPTER XVIII.

PERMUTATIONS—COMBINATIONS—BINOMIAL THEOREM—EXTRACTION OF HIGHER ROOTS.

PERMUTATIONS.

463. ***The Permutations*** of n things, taken r at a time, are the results obtained by arranging the things in every possible order in groups of r each. Thus, the permutations of the letters a, b, c, taken *two* at a time, are

$$ab, \quad ba, \quad ac, \quad ca, \quad bc, \quad cb.$$

The permutations of the same letters taken all at a time, are

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

It is evident that r cannot be greater than n.

464. To find the number of permutations of n things, taken r at a time.

Suppose the n things to be n letters, $a, b, c, d \dots$.

The number of permutations of n letters, taken one at a time, is n.

In order to form all the permutations of n letters, taken two at a time, we must annex to each letter each of the $n-1$ other letters. We thus obtain $n(n-1)$ permutations.

In order to form all the permutations of n letters, taken three at a time, we must annex to each of the permutations, taken two at a time, each of the $n-2$ other letters. We thus obtain $n(n-1)(n-2)$ permutations.

In the same manner it may be shown that the number of permutations of n letters, taken 4 at a time, is $n(n-1)(n-2)(n-3)$.

From these cases it might be *conjectured* that the number of permutations of n letters, taken r at a time, is

$$n(n-1)(n-2)(n-3)\ \ldots\ldots\ (n-r+1).$$

To show that this is true, we employ the method of *mathematical induction.*

Denote the number of permutations of n letters, taken r at a time, by P_r, and suppose for a certain value of r that

$$P_r = n(n-1)(n-2)(n-3)\ \ldots\ldots\ (n-r+1)\ \ .\ \ .\ \ .\ \ (A).$$

Now, in order to form all the permutations of n letters, taken $r+1$ at a time, we must annex to each of the P_r permutations each of the $n-r$ other letters. We thus form

$$n(n-1)(n-2)(n-3)\ \ldots\ldots\ (n-r+1)(n-r)$$

permutations. Hence, denoting the number of permutations of n letters, taken $r+1$ at a time, by P_{r+1}, we have

$$P_{r+1} = n(n-1)(n-2)(n-3)\ \ldots\ldots\ (n-r+1)(n-r),$$

which may be written

$$P_{r+1} = n(n-1)(n-2)(n-3)\ \ldots\ldots\ (n-r+1)[n-(r+1)+1].$$

This equation is of the same form as (A); that is, it may be derived from (A) by simply substituting $r+1$ for r.

If then (A) is true when r is a certain number, it is true when we increase that number by one. But (A) has been shown to be true when $r=3$; it is therefore true when $r=4$; it is therefore true when $r=5$; and so on. Hence the formula must be universally true.

Cor.—If $r=n$, (A) becomes

$$P_n = n(n-1)(n-2)(n-3)\ \ldots\ldots\ 1\ \ .\ \ .\ \ .\ \ (B).$$

That is,

The number of permutations of n things, taken n at a time, is equal to the product of the consecutive numbers from 1 to n inclusive.

For brevity, $n(n-1)(n-2)(n-3) \ldots . 1$ is often denoted by the symbol $\lfloor n$, which is read, *factorial n.*

465. To find the number of permutations of n things, taken n at a time, when some of the things are identical.

Suppose the n things to be n letters; and suppose p of them to be a, q of them to be b, r of them to be c, and the others to be unlike.

Denote the required number of permutations by N. If in any one of these N permutations the p letters a were changed into p new letters different from each other, and also different from all the other letters contained in the permutation, then, without changing the situation of the other letters, we could from the single permutation form $\lfloor p$ different permutations; therefore the whole number of permutations would be $N \times \lfloor p$. In like manner, if the q letters b were changed into q new letters different from each other, and also different from all the other letters contained in the $N \times \lfloor p$ permutations, we could form $N \times \lfloor p \times \lfloor q$ permutations; and if the r letters c were also changed in the same way, the number of permutations would be $N \times \lfloor p \times \lfloor q \times \lfloor r$. But this number must be equal to the number of permutations of n dissimilar things, taken n at a time; hence,

$$N \times \lfloor p \times \lfloor q \times \lfloor r = \lfloor n;$$

whence,
$$N = \frac{\lfloor n}{\lfloor p \times \lfloor q \times \lfloor r} \quad . \quad . \quad . \quad (C).$$

466. ***PROBLEMS.***

1. How many different permutations may be formed of 8 letters, taken 5 at a time? *Ans.* 6720.

2. How many different permutations may be formed of all the letters of the alphabet, taken 4 at a time? *Ans.* 358800.

3. How many different permutations may be made of 6 things, taken 6 at a time? *Ans.* 720.

4. How many different numbers may be formed with the five figures, 5, 4, 3, 2, 1, each figure occurring once, and only once, in each number? *Ans.* 120.

5. How many different permutations may be made of the letters in the word *Longitude*, taken all together? *Ans.* 362880.

6. How many different permutations may be made of the letters in the word *Caraccas*, taken all together? *Ans.* 1120.

7. How many different permutations may be made of the letters in the word *Heliopolis*, taken all together? *Ans.* 453600.

8. How many different permutations may be made of the letters in the word *Ecclesiastical*, taken all together?

Ans. 454053600.

9. What value must n have in order that the number of permutations of n things, taken 4 at a time, may be equal to 12 times the number of permutations of n things, taken 2 at a time?

Ans. $n = 6$.

COMBINATIONS.

467. ***The Combinations*** of n things, taken r at a time, are the results obtained by arranging the things in as many different groups of r each as possible, without regarding the order in which the things are placed. Thus, the combinations of the letters a, b, c, taken *two* at a time, are

$$ab, \quad ac, \quad bc.$$

It will be observed that if the letters be regarded as *factors*, the combinations which may be formed by taking r at a time will constitute all the different products of the r^{th} degree, of which the letters are capable.

468. To find the number of combinations of n things, taken r at a time.

Denote the number of combinations of n things, taken r at a time, by C_r, and the number of permutations of n things, taken r at a time, by P_r.

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It is evident that all of the P_r permutations can be formed by subjecting the r letters of each of the C_r combinations to all the *permutations* of which these letters are susceptible, when taken r at a time. Now, the number of permutations of r letters, taken r at a time, is $\underline{|r}$ (**464,** Cor.); therefore the number of permutations of n letters, taken r at a time, is $C_r \underline{|r}$; hence,

$$C_r \times \underline{|r} = P_r;$$

whence,
$$C_r = \frac{P_r}{\underline{|r}}.$$

But $P_r = n(n-1)(n-2)(n-3) \ldots\ldots (n-r+1)$ (**464**);

$$\therefore \quad C_r = \frac{n(n-1)(n-2)(n-3) \ldots\ldots (n-r+1)}{\underline{|r}} \quad . \quad . \quad . \quad (D).$$

469. *The number of combinations of n things, taken r at a time, is equal to the number of combinations of n things, taken $n-r$ at a time.*

Denote the number of combinations of n things, taken $n-r$ at a time, by C_{n-r}; then by (D),

$$C_{n-r} = \frac{n(n-1)(n-2)(n-3) \ldots\ldots [n-(n-r)+1]}{\underline{|n-r}}$$

$$= \frac{n(n-1)(n-2)(n-3) \ldots\ldots (r+1)}{\underline{|n-r}} \quad . \quad . \quad . \quad (1).$$

Multiplying both terms of the second member of (1) by $\underline{|r}$,

$$C_{n-r} = \frac{\underline{|n}}{\underline{|r} \times \underline{|n-r}} \quad . \quad . \quad . \quad (2).$$

Multiplying both terms of the second member of (D) by $\underline{|n-r}$,

$$C_r = \frac{\underline{|n}}{\underline{|r} \times \underline{|n-r}} \quad . \quad . \quad . \quad (3);$$

$$\therefore \quad C_r = C_{n-r} \quad . \quad . \quad . \quad (4).$$

470. ***PROBLEMS.***

1. Find the number of different products that can be formed with the numbers 1, 2, 3, 4, 5, taken 2 at a time. *Ans.* 10.

2. What value must n have in order that the number of permutations of n things, taken 5 at a time, may be equal to 120 times the number of combinations of n things, taken 3 at a time?

Ans. 8.

3. When n is even, what value must r have, in order that C_r may be the greatest possible? *Ans.* $r = \frac{n}{2}$.

4. When n is odd, what value must r have, in order that C_r may be the greatest possible? *Ans.* $r = \frac{n \pm 1}{2}$.

5. From a company of soldiers numbering 96 men a picket of 10 is to be selected; in how many ways can it be done so as always to include a particular man? *Ans.* $\dfrac{|\underline{95}}{|\underline{9} \times |\underline{86}}$.

6. From a company of soldiers numbering 96 men, a picket of 10 is to be selected; in how many ways can it be done so as always to exclude a particular man? *Ans.* $\dfrac{|\underline{95}}{|\underline{10} \times |\underline{85}}$.

THE BINOMIAL FORMULA.

471.

$$(x+a)(x+b) = x^2 \begin{array}{r|} +a \\ +b \end{array}\, x + ab.$$

$$(x+a)(x+b)(x+c) = x^3 \begin{array}{r|} +a \\ +b \\ +c \end{array}\, x^2 \begin{array}{r|} +ab \\ +ac \\ +bc \end{array}\, x + abc.$$

$$(x+a)(x+b)(x+c)(x+d) = x^4 \begin{array}{r|} +a \\ +b \\ +c \\ +d \end{array}\, x^3 \begin{array}{r|} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\, x^2 \begin{array}{r|} +abc \\ +abd \\ +acd \\ +bcd \end{array}\, x + abcd.$$

In each of these identities we observe the following laws:

1. *The number of terms in the second member is one greater than the number of binomial factors in the first member.*

2. *The exponent of x in the first term of the second member is equal to the number of binomial factors, and in each of the succeeding terms the exponent of x is one less than in the preceding term.*

3. *The coefficient of the first term of the second member is unity, the coefficient of the second term is the sum of the second terms of the binomial factors; the coefficient of the third term is the sum of all the products of the second terms of the binomial factors, taken two at a time; the coefficient of the fourth term is the sum of all the products of the second terms of the binomial factors, taken three at a time; and so on: the last term is the product of all the second terms of the binomial factors.*

472. That the laws stated in the preceding Article are general may be shown as follows:

Suppose the laws to be true in the case of n binomials, $x+a$, $x+b$, $x+c$ .... $x+k$; that is, suppose

$$(x+a)(x+b)(x+c)\ \ldots\ (x+k) = x^n + P_1x^{n-1} + P_2x^{n-2} + P_3x^{n-3} \ldots + P_n \quad \ldots \quad (1),$$

in which $P_1 =$ the sum of the terms a, b, c .... k,

$P_2 =$ the sum of the products of these terms, taken two at a time,

$P_3 =$ the sum of the products of these terms, taken three at a time,

. . . .

$P_n =$ the product of all these terms.

Multiplying both members of (1) by $x+l$,

$$(x+a)(x+b)(x+c)\ \ldots\ (x+k)(x+l) =$$

$$x^{n+1} + \left.\begin{matrix} P_1 \\ +l \end{matrix}\right| x^n + \left.\begin{matrix} P_2 \\ +P_1l \end{matrix}\right| x^{n-1} + \left.\begin{matrix} P_3 \\ +P_2l \end{matrix}\right| x^{n-2} \ \ldots + P_nl \quad \ldots \quad (2).$$

Now $\quad P_1 + l = a + b + c \; . \; . \; . \; . + k + l$
$= \text{the sum of all the terms } a, b, c, \ldots . k, l;$

$P_2 + P_1 l = P_2 + (a + b + c \; . \; . \; . \; . + k) l$
$= \text{the sum of the products of all the terms } a, b, c, \ldots . k, l, \text{ taken two at a time};$

$P_3 + P_2 l = P_3 + (ab + ad + ac + bc + bd + cd \; . \; . \; .) l$
$= \text{the sum of the products of all the terms } a, b, c, \ldots . k, l, \text{ taken three at a time};$

. . . .

$P_n l = \text{the product of all the terms } a, b, c, \ldots . k, l.$

The law of the exponents in (2) is also the same as in (1).

Hence, if the laws are true when n factors are used, they will be true when $n+1$ factors are used. But they have been shown to be true when $n=4$; therefore they are true when $n=5$; and so on. Hence the laws must be true universally.

473. The number of terms in P_1 is obviously n; the number of terms in P_2 is equal to the number of combinations of n things, taken two at a time, that is, $\frac{n(n-1)}{\lfloor 2}$ (**468**); the number of terms in P_3 is equal to the number of combinations of n things, taken three at a time, that is, $\frac{n(n-1)(n-2)}{\lfloor 3}$; and so on.

Now suppose $a = b = c = \ldots . k$; then $P_1 = na$, $P_2 = \frac{n(n-1)}{\lfloor 2} a^2$, $P_3 = \frac{n(n-1)(n-2)}{\lfloor 3} a^3$, and so on. Under this hypothesis, (1) of Art. **472** becomes

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{\lfloor 2} a^2 x^{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3} a^3 x^{n-3} + \ldots . + \frac{n(n-1)}{\lfloor 2} a^{n-2} x^2 + na^{n-1} x + a^n.$$

This is the ***Binomial Formula.*** The second member of this formula is called the ***Expansion*** or ***Development*** of $(x+a)^n$, and when we put this expansion or development in the place of $(x+a)^n$ we are said to expand or develop $(x+a)^n$.

474. The coefficient of the product of the powers of a and x in any term of the expansion of $(x+a)^n$ is called the *coefficient* of that term. Thus, the coefficient of the third term of the expansion of $(x+a)^n$ is $\dfrac{n(n-1)}{\lfloor 2}$.

475. The first letter in an expression of the form of $(x+a)^n$ is called the *leading letter.*

476. ***Another Proof of the Binomial Formula.***—We can verify the Binomial Formula by trial for small values of n as 2, 3, 4.

Assume

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{\lfloor 2} a^2x^{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3} a^3x^{n-3}$$

$$+ \dots + a^n \quad . \quad . \quad . \quad (1).$$

Multiplying both members of (1) by $x+a$ and reducing,

$$(x+a)^{n+1} = x^{n+1} + (n+1)ax^n + \frac{(n+1)n}{\lfloor 2} a^2x^{n-1}$$

$$+ \frac{(n+1)n(n-1)}{\lfloor 3} a^3x^{n-2} + \dots + a^{n+1} \quad . \quad . \quad . \quad (2);$$

that is, the expansion of $(x+a)^{n+1}$ is of the same form as that of $(x+a)^n$. Hence, if the Binomial Formula is true for any exponent, it is true when the exponent is increased by unity. But the formula *is* true when $n=4$; it is therefore true when $n=5$; it is therefore true when $n=6$; and so on. Hence the Binomial Formula is true for any positive integral exponent.

Cor. 1.—*If we multiply the coefficient of any term in the expansion of* $(x+a)^n$ *by the exponent of* x *in that term, and divide the product by the number obtained by adding* 1 *to the exponent of* a *in the same term, the quotient will be the coefficient of the succeeding term.*

Cor. 2.—*The sum of the exponents of* a *and* x *in any term of the expansion of* $(x+a)^n$ is equal to n.

477. To find the sum of the coefficients in the expansion of $(x+a)^n$.

The formula

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{\underline{|2}}a^2x^{n-2} + \frac{n(n-1)(n-2)}{\underline{|3}}a^3x^{n-3}$$
$$+ \ldots\ldots + a^n$$

is true for all values of x and a, and the coefficients are independent of x and a. Suppose $x = a = 1$; we then have

$$2^n = 1 + n + \frac{n(n-1)}{\underline{|2}} + \frac{n(n-1)(n-2)}{\underline{|3}} + \ldots\ldots + 1.$$

That is, the sum of the coefficients in the expansion of $(x+a)^n$ is 2^n.

478. To find the r^{th} term of the expansion of $(x+a)^n$.

The exponent of a in any term of the expansion of $(x+a)^n$ is one less than the number of that term; hence the exponent of a in the r^{th} term is $r-1$. The sum of the exponents of a and x in any term is n; hence the exponent of x in the r^{th} term is $n-r+1$. The coefficient of the second term is equal to the number of combinations of n things, taken *one* at a time; the coefficient of the third term is equal to the number of combinations of n things, taken *two* at a time; and so on; hence the coefficient of the r^{th} term is equal to the number of combinations of n things taken $r-1$ at a time; that is,

$$\frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-r+2)}{\underline{|r-1}} \quad \textbf{(468)}.$$

Therefore, denoting the r^{th} term by T_r, we have

$$T_r = \frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-r+2)}{\underline{|r-1}}a^{r-1}x^{n-r+1}.$$

It should be observed that r cannot be greater than $n+1$.

479. *In the expansion of* $(x+a)^n$ *the coefficient of the* r^{th} *term from the beginning is equal to the coefficient of the* r^{th} *term from the end.*

The coefficient of the r^{th} term from the beginning is

$$\frac{n(n-1)(n-2)(n-3)\ \ldots\ (n-r+2)}{\lfloor r-1} \quad (478).$$

Multiplying both numerator and denominator by $\lfloor n-r+1$, this becomes

$$\frac{\lfloor n}{\lfloor r-1 \times \lfloor n-r+1}.$$

The r^{th} term from the end is the $(n-r+2)^{th}$ term from the beginning, and its coefficient is

$$\frac{n(n-1)(n-2)(n-3)\ \ldots\ [n-(n-r+2)+2]}{\lfloor n-r+1} \quad (478),$$

which becomes by reduction

$$\frac{n(n-1)(n-2)(n-3)\ \ldots\ r}{\lfloor n-r+1}.$$

Multiplying both numerator and denominator by $\lfloor r-1$, this becomes

$$\frac{\lfloor n}{\lfloor r-1 \times \lfloor n-r+1}.$$

480. To obtain the ***Expansion of*** $(x-a)^n$, it is sufficient to put $-a$ in the place of $+a$ in the expansion of $(x+a)^n$. The terms which contain the odd powers of $-a$ will be negative, and the terms which contain the even powers of $-a$ will be positive. Hence,

$$(x-a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{\lfloor 2}a^2x^{n-2} - \frac{n(n-1)(n-2)}{\lfloor 3}a^3x^{n-3}$$
$$+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}a^4x^{n-4}\ \ldots\ldots$$

481. *EXAMPLES.*

1. Expand $(a+b)^5$.

First Solution.—In the expansion of $(a+b)^5$ the powers of a are

$$a^5,\quad a^4,\quad a^3,\quad a^2,\quad a^1,\quad a^0;$$

the powers of b are $b^0,\ b^1,\ b^2,\ b^3,\ b^4,\ b^5;$

and the coefficients are $1,\ 5,\ 10,\ 10,\ 5,\ 1$ (**476,** Cor. 1).

$$\therefore\quad (a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Second Solution.—The literal parts of the terms of the expansion of $(a+b)^5$ are

$$a^5,\quad a^4b,\quad a^3b^2,\quad a^2b^3,\quad ab^4,\quad b^5,$$

and the coefficients are $1,\ 5,\ 10,\ 10,\ 5,\ 1;$

$$\therefore\quad (a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Third Solution.—Substituting a for x, b for a, and 5 for n, in the Binomial Formula, we have

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

2. Expand $(2x-3a)^4$.

Powers of $2x$,	$16x^4$,	$8x^3$,	$4x^2$,	$2x$,	$(2x)^0$;
Powers of $-3a$,	$(-3a)^0$,	$-3a$,	$9a^2$,	$-27a^3$,	$81a^4$;
Coefficients,	1,	4,	6,	4,	1;

$$\therefore\quad (2x-3a)^4 = 16x^4 - 96ax^3 + 216a^2x^2 - 216a^3x + 81a^4.$$

3. Expand $(a+b+c+d)^3$.

Put $x=a+b$ and $y=c+d$; then

$$(a+b+c+d)^3 = (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Substituting for x and y their values,

$$\begin{aligned}(a+b+c+d)^3 &= (a+b)^3 + 3(a+b)^2(c+d) + 3(a+b)(c+d)^2 + (c+d)^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3a^2d\\ &\quad + 6abd + 3b^2d + 3ac^2 + 6acd + 3ad^2 + 3bc^2 + 6bcd\\ &\quad + 3bd^2 + c^3 + 3c^2d + 3cd^2 + d^3.\end{aligned}$$

4. Find the 5th term of the expansion of $(a+b)^{15}$.

Substituting a for x, b for a, 15 for n, and 5 for r in the formula of Art. **478**, we have

$$T_5 = \frac{15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4} b^4 a^{11} = 1365 b^4 a^{11}.$$

5. Expand $(a-b)^5$.

Ans. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

6. Expand $(1+c)^6$.

Ans. $1 + 6c + 15c^2 + 20c^3 + 15c^4 + 6c^5 + c^6$.

7. Expand $(x+y)^7$.

Ans. $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$.

8. Expand $(a^2 - 1)^8$.

Ans. $a^{16} - 8a^{14} + 28a^{12} - 56a^{10} + 70a^8 - 56a^6 + 28a^4 - 8a^2 + 1$.

9. Expand $(a-c)^9$.

Ans. $a^9 - 9a^8c + 36a^7c^2 - 84a^6c^3 + 126a^5c^4 - 126a^4c^5 + 84a^3c^6 - 36a^2c^7 + 9ac^8 - c^9$.

10. Expand $(1+ax)^5$.

Ans. $1 + 5ax + 10a^2x^2 + 10a^3x^3 + 5a^4x^4 + a^5x^5$.

11. Expand $(3a+2c)^5$.

Ans. $243a^5 + 810a^4c + 1080a^3c^2 + 720a^2c^3 + 240ac^4 + 32c^5$.

12. Expand $\left(5 - \frac{x}{6}\right)^6$.

Ans. $15625 - 3125x + \frac{3125}{12}x^2 - \frac{625}{54}x^3 + \frac{125}{432}x^4 - \frac{5x^5}{1296} + \frac{x^6}{46656}$.

13. Expand $(a^2 - ab + b^2)^4$.

Ans. $a^8 - 4a^7b + 10a^6b^2 - 16a^5b^3 + 19a^4b^4 - 16a^3b^5 + 10a^2b^6 - 4ab^7 + b^8$.

14. Find the middle term of the expansion of $(a+x)^{10}$.

Ans. $252a^5x^5$.

15. Find the middle term of the expansion of $\left(\frac{3a}{4} + \frac{4r}{5}\right)^4$.

Ans. $\frac{54}{25}a^2r^2$.

16. Find the 2001st term of the expansion of $\left(a^{\frac{3}{10}} + x^{\frac{3}{10}}\right)^{2002}$.

Ans. $2003001a^{\frac{6}{10}}x^{600}$.

THE n^{th} ROOT OF QUANTITIES.

482. To find the n^{th} root of a polynomial.

Find the n^{th} root of $x^n + nax^{n-1} + \frac{n(n-1)}{\lfloor 2} a^2x^{n-2} \ldots . + a^n$.

$$x^n + nax^{n-1} + \frac{n(n-1)}{\lfloor 2} a^2x^{n-2} \ldots . + a^n \,\big|\, x + a$$

$$(x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{\lfloor 2} a^2x^{n-2} \ldots . + a^n$$

Arrange the terms according to the descending powers of x. The n^{th} root of the first term, x^n, is x, which is the first term of the required root. The second term of the root may be found by dividing the second term of the given polynomial by nx^{n-1}.

If there were more terms in the root, we should proceed with $x + a$ as we did with x.

RULE.

I. *Arrange the given polynomial according to the powers of one of its letters.*

II. *Extract the n^{th} root of the first term; the result will be the first term of the required root. Subtract the n^{th} power of this term from the given polynomial.*

III. *Divide the first term of the remainder by n times the $(n-1)^{th}$ power of the first term of the root; the quotient will be the second term of the root. Subtract the n^{th} power of the sum of the first and second terms of the root from the given polynomial.*

IV. *Take n times the $(n-1)^{th}$ power of the sum of the first and second terms of the root for a second divisor. Divide the first term of the second remainder by the first term of the second divisor; the quotient will be the third term of the root. Subtract the n^{th} power of the sum of the first, second, and third terms of the root from the given polynomial.*

V. *Proceed in this manner until all the terms of the root have been found.*

COR.—If the root contains only two terms, it may be obtained by extracting the n^{th} root of the extreme terms of the arranged polynomial, and placing the proper sign between the results. Thus, the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$, and the cube root of $a^3 - 3a^2b + 3ab^2 - b^3$ is $a - b$.

EXAMPLES.

1. Find the fourth root of $16a^4 - 96a^3x + 216a^2x^2 + 81x^4 - 216ax^3$. *Ans.* $2a - 3x$.

2. Find the fifth root of $80a^3 + 32a^5 - 80a^4 - 40a^2 + 10a - 1$. *Ans.* $2a - 1$.

3. Find the fourth root of $336a^5 + 81a^8 - 216a^7 - 56a^4 + 16 - 224a^3 + 64a$. *Ans.* $3a^2 - 2a - 2$.

4. Find the fourth root of $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$. *Ans.* $a - b$.

5. Find the fifth root of $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$. *Ans.* $a + b$.

6. Find the sixth root of $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$. *Ans.* $a - 1$.

7. Find the seventh root of $a^7 + 7a^6 + 21a^5 + 35a^4 + 35a^3 + 21a^2 + 7a + 1$. *Ans.* $a + 1$.

8. Find the eighth root of $x^8 - 8x^7 + 28x^6 - 84x^5 + 105x^4 - 84x^3 + 28x^2 - 8x + 1$. *Ans.* $x - 1$.

483. To find the n^{th} root of a number.

For a reason similar to that given in Art. **271,** we separate the given number into periods of n figures each, beginning with units. The n^{th} root of the greatest n^{th} power contained in the period on the left will be the first figure of the root. If we subtract the n^{th} power of the first figure from the given number, and divide the remainder by n times the $(n-1)^{th}$ power of that figure, regarding its local value, the quotient will be the second figure of the root, or a figure too large. The result may be tested by subtracting the n^{th} power of the number represented by the

first and second figures of the root from the given number. If there are additional figures in the root, they may be found in the same manner.

EXAMPLES.

1. Find the fifth root of 33554432.

$$\begin{array}{rl} & 335,54432(30+2=32 \\ 30^5 = & 243\,00000 \\ 5 \times 30^4 = 4050000) & \overline{92\,54432} \\ 32^5 = & 33554432 \end{array}$$

2. Find the fourth root of 79502005521. *Ans.* 531.
3. Find the fourth root of 75450765.3376. *Ans.* 93.2.
4. Find the fourth root of 2526.88187761. *Ans.* 7.09.
5. Find the fifth root of 418227202051. *Ans.* 211.
6. Find the seventh root of 34359738368. *Ans.* 32.

484. SYNOPSIS FOR REVIEW.

CHAPTER XVIII.

- PERMUTATIONS
  - n THINGS TAKEN r AT A TIME.
  - n THINGS TAKEN n AT A TIME.
  - n THINGS TAKEN n AT A TIME, WHEN SOME ARE IDENTICAL.
- COMBINATIONS
  - n THINGS TAKEN r AT A TIME.
  - INTERPRET THE EQUATION $C_r = C_{n-r}$.
- BINOMIAL FORMULA.
  - PRODUCT OF n BINOMIALS WHOSE FIRST TERMS ARE IDENTICAL AND WHOSE SECOND TERMS ARE DIFFERENT.
  - GENERAL LAWS. 1, 2, 3.
  - BINOMIAL FORMULA.
  - COEFFICIENT OF A TERM OF THE EXPANSION OF $(x+a)^n$.
  - LEADING LETTER.
  - ANOTHER PROOF OF BINOMIAL FORMULA. *Cor. 1, 2.*
  - TO FIND THE SUM OF COEFFICIENTS IN THE EXPANSION OF $(x+a)^n$.
  - TO FIND THE r^{th} TERM OF THE EXPANSION OF $(x+a)^n$.
  - EXPANSION OF $(x-a)^n$.
- EXTRACTION OF HIGHER ROOTS.
  - TO FIND THE n^{th} ROOT OF A POLYNOMIAL. *Rule.* Cor.
  - TO FIND THE n^{th} ROOT OF A NUMBER.

CHAPTER XIX.

IDENTICAL EQUATIONS.

PROPERTIES OF IDENTICAL EQUATIONS.

485. *If the equation*

$$A + Bx + Cx^2 + Dx^3 + \text{etc.} = A' + B'x + C'x^2 + D'x^3 + \text{etc.},$$

in which A, B, C, D, etc., A′, B′, C′, D′, etc., *are finite quantities independent of* x, *is an identity, the coefficients of the like powers of* x *are equal to each other.*

Since this equation is true for *every* value that may be assigned to x (**178**), it must be true when $x = 0$. But when $x = 0$, all the terms disappear except A and A′, and the equation becomes

$$A = A'.$$

Dropping A from one member of the given equation, and A from the other,

$$Bx + Cx^2 + Dx^3 + \text{etc.} = B'x + C'x^2 + D'x^3 + \text{etc.}$$

Dividing both members of this equation by x,

$$B + Cx + Dx^2 + \text{etc.} = B' + C'x + D'x^2 + \text{etc.}$$

Making $x = 0$, as before, this equation becomes

$$B = B'.$$

In like manner it may be shown that

$$C = C',$$

$$D = D', \text{ etc.}$$

486. *If the equation*

$$A + Bx + Cx^2 + Dx^3 + \text{etc.} = 0$$

is an identity, each of the coefficients A, B, C, D, etc., *is equal to zero.*

Since this equation is true for *every* value that may be assigned to x, it must be true when $x = 0$. But when $x = 0$, the given equation becomes

$$A = 0.$$

Dropping A from the given equation, and dividing the result by x,

$$B + Cx + Dx^2 + \text{etc.} = 0.$$

Making $x = 0$, as before, this equation becomes

$$B = 0.$$

In like manner it may be shown that

$$C = 0,$$

$$D = 0, \text{ etc.}$$

487. ***Undetermined Coefficients*** are such as are unknown in an assumed identity. Thus, if we assume $(x + a)^3 = Ax^3 + Bx^2 + Cx + D$ to be *identically* true, A, B, C, and D are undetermined coefficients.

DECOMPOSITION OF RATIONAL FRACTIONS.

488. ***To Decompose a Rational Fraction*** is to separate it into fractions whose sum is equal to the given fraction and the product of whose denominators is equal to the given denominator. The parts into which the given fraction is separated are called ***Partial Fractions.***

EXAMPLES.

1. Separate $\dfrac{8x - 31}{x^2 - 7x + 10}$ into partial fractions.

The factors of the denominator are $x - 5$ and $x - 2$; hence

the denominator of one of the partial fractions is $x-5$, and that of the other is $x-2$.

$$\text{Assume} \quad \frac{8x-31}{x^2-7x+10} = \frac{A}{x-5} + \frac{B}{x-2} \quad . \quad . \quad . \quad (1).$$

Since the first member is the sum of the two fractions in the second member, this equation is an *identity*.

Clearing (1) of fractions and uniting similar terms,

$$8x-31=(A+B)x-(2A+5B) \quad . \quad . \quad . \quad (2).$$

$$\therefore \qquad \left\{\begin{array}{l} A + \ B = \ 8 \\ 2A + 5B = 31 \end{array}\right\} \ \textbf{(485)}.$$

whence, $\qquad A = 3 \quad \text{and} \quad B = 5.$

Substituting 3 for A and 5 for B in (1),

$$\frac{8x-31}{x^2-7x+10} = \frac{3}{x-5} + \frac{5}{x-2}.$$

2. Separate $\dfrac{7x^2+x}{(x+1)(2x-1)}$ into partial fractions.

$$\text{Assume} \quad \frac{7x^2+x}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1} \quad . \quad . \quad . \quad (1).$$

Clearing of fractions and uniting similar terms,

$$7x^2 + x = (2A+B)x + B - A \quad . \quad . \quad . \quad (2).$$

The coefficient of x^2 in the second member of (2) is 0;

$$\therefore \qquad 7 = 0 \quad \textbf{(485)},$$

which is absurd. Hence the given fraction cannot be separated into partial fractions having numerators independent of x.

$$\text{Assume} \quad \frac{7x^2+x}{(x+1)(2x-1)} = \frac{Ax}{x+1} + \frac{Bx}{2x-1} \quad . \quad . \quad . \quad (3).$$

Clearing of fractions and uniting similar terms,

$$7x^2 + x = (2A+B)x^2 + (B-A)x \quad . \quad . \quad . \quad (4).$$

Equating the coefficients of like powers of x in (4),

$$\left\{\begin{array}{c} 2A + B = 7 \\ B - A = 1 \end{array}\right\};$$

whence, $A = 2$ and $B = 3$.

Substituting 2 for A and 3 for B in (3),

$$\frac{7x^2 + x}{(x+1)(2x-1)} = \frac{2x}{x+1} + \frac{3x}{2x-1}.$$

From this example we learn that if we assume an impossible form for the partial fractions, the fact will be made apparent by some absurdity in the equations of condition (**179**).

Separate each of the following fractions into its partial fractions:

3. $\dfrac{7x - 24}{x^2 - 9x + 14}$. *Ans.* $\dfrac{5}{x-7} + \dfrac{2}{x-2}$.

4. $\dfrac{20x + 2}{2x^2 + 3x - 20}$. *Ans.* $\dfrac{8}{2x-5} + \dfrac{6}{x+4}$.

5. $\dfrac{6x^2 - 22x + 18}{(x-1)(x^2 - 5x + 6)}$. *Ans.* $\dfrac{1}{x-1} + \dfrac{2}{x-2} + \dfrac{3}{x-3}$.

6. $\dfrac{x + 2}{x^3 - x}$. *Ans.* $\dfrac{\frac{1}{2}}{x+1} + \dfrac{\frac{3}{2}}{x-1} - \dfrac{2}{x}$.

7. $\dfrac{10}{x^4 - 13x^2 + 36}$. *Ans.* $\dfrac{\frac{1}{2}}{x+2} - \dfrac{\frac{1}{2}}{x-2} - \dfrac{\frac{1}{3}}{x+3} + \dfrac{\frac{1}{3}}{x-3}$.

8. $\dfrac{3x^3 + 5x^2 - 2x}{x^2 - 1}$. *Ans.* $\dfrac{3x^2}{x-1} + \dfrac{2x}{x+1}$.

489. SYNOPSIS FOR REVIEW.

CHAP. XIX. IDENTICAL EQUATIONS.

- PROPERTIES OF IDENTICAL EQUATIONS. { **485.** **486.** }
- UNDETERMINED COEFFICIENTS.
- DECOMPOSITION OF RATIONAL FRACTIONS. PARTIAL FRACTIONS.

CHAPTER XX.

SERIES.

GENERAL DEFINITIONS.

490. ***A Series*** is a succession of quantities, each of which, except the first, or a certain number of the first, may be derived from the preceding one, or a certain number of the preceding ones, by a fixed law called the ***Law of the Series.*** Thus,

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9,$$

is a series, the law of which is that each quantity, except the first, is derived from the preceding one by adding unity to it.

491. ***The Terms*** of a series are the quantities of which the series consists.

492. ***A Finite Series*** is one which, by its law of formation, can have only a *finite* number of terms. Such a series is said to *terminate.* Thus, the expansion of $(x + a)^n$, when n is a positive integer, is a finite series.

493. ***An Infinite Series*** is one which, by its law of formation, may have an *infinite* number of terms. Such a series is said *not to terminate.* Thus,

$$1, \ \frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{8}, \ \frac{1}{16}, \ \frac{1}{32}, \ \frac{1}{64}, \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot$$

is an infinite series.

494. ***A Converging Series*** is an infinite series, the sum of the first n terms of which cannot numerically exceed some finite quantity, however great n may be. Thus,

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots\ldots$$

is a converging series, for the sum of its first n terms cannot exceed 2, however great n may be.

495. ***A Diverging Series*** is an infinite series, the sum of the first n terms of which can be made numerically greater than any finite quantity by taking n sufficiently great. Thus,

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ \ldots\ldots$$

is a diverging series.

ARITHMETICAL PROGRESSION.

496. ***An Arithmetical Progression,*** or ***a Progression by Difference,*** is a series in which the difference between the first and second terms is equal to the difference between any other two consecutive terms. Thus, 1, 3, 5, 7, 9 is an arithmetical progression.

An arithmetical progression is sometimes called an ***Arithmetical Series.***

For brevity we shall sometimes use A. P. for the phrase *arithmetical progression.*

497. ***The Extremes*** of an A. P. are the first term and the last term; the other terms are the ***Means.***

498. ***The Common Difference*** of an A. P. is the remainder obtained by subtracting any term from the one which follows it. Thus, in the progression 1, 3, 5, 7, 9, the common difference is 2.

499. ***An Increasing A. P.*** is one in which the common difference is *positive.* Thus, 1, 2, 3, 4, 5 is an increasing A. P.

500. ***A Decreasing A. P.*** is one in which the common difference is *negative.* Thus, 9, 7, 5, 3, 1 is a decreasing A. P.

501. *Notation.*—In treating arithmetical progressions we shall use the following notation:

$a =$ the first term of the progression,
$l =$ the last or n^{th} term,
$d =$ the common difference,
$n =$ the number of terms,
$s =$ the sum of all the terms.

Thus, in the A. P. $\quad 1, \; 3, \; 5, \; 7, \; 9,$

$a = 1, \; l = 9, \; d = 2, \; n = 5, \; s = 25.$

502. To find l when a, d, and n are given.

The first term is a, the second term is $a + d$, the third term is $a + 2d$, the fourth term is $a + 3d$, and so on; hence the n^{th} term is $a + (n - 1)d$; that is,

$$l = a + (n - 1)d.$$

503. To find s when a, l, and n are given.

$$s = a + (a + d) + (a + 2d) + (a + 3d) + \ldots\ldots + l \quad . \; . \; . \quad (1).$$

Inverting the order of terms in the second member of (1),

$$s = l + (l - d) + (l - 2d) + (l - 3d) + \ldots\ldots + a \quad . \; . \; . \quad (2).$$

Adding (1) and (2),

$$2s = (a + l) + (a + l) + (a + l) + \ldots\ldots + (a + l) = n(a + l) \; . \; . \; . \; (3);$$

whence,
$$s = \frac{n}{2}(a + l) \quad . \; . \; . \quad (4).$$

504. *In an A. P. the sum of any two terms equidistant from the extremes is equal to the sum of the extremes.*

Let x denote a term which has m terms before it, and y a term which has m terms after it; then

$$\left. \begin{aligned} x &= a + md \\ y &= l - md \end{aligned} \right\} (502);$$

whence,
$$x + y = a + l.$$

505. To insert any number of arithmetical means between two given quantities.

Let a and b be the given quantities, and let it be required to insert m arithmetical means between them; that is, let it be required to form an A. P. whose extremes are a and b and the number of whose terms is $m+2$.

Substituting b for l and $m+2$ for n in the formula of Art. **502**, we have

$$b = a + (m+1)\,d;$$

whence,
$$d = \frac{b-a}{m+1}.$$

By adding the common difference to a we obtain the second term; by adding it to the second term we obtain the third; and so on.

Example.—Insert 10 arithmetical means between 5 and 38.

$$d = \frac{38-5}{10+1} = 3;$$

hence the required progression is

$$5,\ 8,\ 11,\ 14,\ 17,\ 20,\ 23,\ 26,\ 29,\ 32,\ 35,\ 38.$$

506. To find any two of the quantities a, l, d, n, and s, when the three others are given.

The group
$$\left\{\begin{array}{l} l = a + (n-1)\,d \\ s = \dfrac{n}{2}(a+l) \end{array}\right\}$$

contains the five quantities a, l, d, n, and s; hence any two of them may be found when the three others are given.

The ten cases are given in the following table as an exercise for the student.

Each case is an example of two simultaneous equations with two unknown quantities.

NO.	GIVEN.	TO FIND.	RESULTING FORMULÆ.
1.	a, d, n	l, s	$l=a+(n-1)d,\ s=\frac{n}{2}[2a+(n-1)d]$.
2.	a, d, l	n, s	$n=\frac{l-a}{d}+1,\quad s=\frac{(l+a)(l-a+d)}{2d}$.
3.	a, d, s	n, l	$\begin{cases} n=\frac{d-2a\pm\sqrt{(2a-d)^2+8ds}}{2d}, \\ l=-\frac{d}{2}\pm\sqrt{2ds+\left(a-\frac{d}{2}\right)^2}. \end{cases}$
4.	a, n, l	d, s	$d=\frac{l-a}{n-1},\quad s=\frac{n}{2}(a+l)$.
5.	a, n, s	d, l	$d=\frac{2(s-an)}{n(n-1)},\quad l=\frac{2s}{n}-a$.
6.	a, l, s	d, n	$d=\frac{l^2-a^2}{2s-a-l},\quad n=\frac{2s}{a+l}$.
7.	d, n, l	a, s	$a=l-(n-1)d,\ s=\frac{n}{2}[2l-(n-1)d]$.
8.	d, n, s	a, l	$a=\frac{2s-n(n-1)d}{2n},\ l=\frac{2s+n(n-1)d}{2n}$.
9.	d, l, s	a, n	$\begin{cases} a=\frac{1}{2}\left[d\pm\sqrt{(2l+d)^2-8ds}\right], \\ n=\frac{2l+d\pm\sqrt{(2l+d)^2-8ds}}{2d}. \end{cases}$
10.	l, n, s	a, d	$a=\frac{2s}{n}-l,\quad d=\frac{2(ln-s)}{n(n-1)}$.

507. ***PROBLEMS.***

1. The first term of an A. P. is 5, the common difference is 3, and the number of terms is 24. Find the last term and the sum of all the terms.

We have given $a=5,\ d=3,\ n=24$;

$$\therefore \qquad l=5+(24-1)3=74,$$

and $$s=\frac{24}{2}[10+(24-1)3]=948 \quad (\mathbf{506},\ 1).$$

After finding the value of l, we might have found the value of s from the formula $s = \frac{n}{2}(a + l)$. Thus,

$$s = \frac{24}{2}(5 + 74) = 948.$$

2. Given $a = 15$, $d = -2$, and $s = 60$, to find l and n.

$$l = -\frac{-2}{2} \pm \sqrt{2(-2)\,60 + \left(15 - \frac{-2}{2}\right)^2} = 5 \text{ or } -3,$$

and

$$n = \frac{-2 - 30 \pm \sqrt{(30 + 2)^2 + 8(-2)\,60}}{2(-2)} = 6 \text{ or } 10\ (\mathbf{506}, 3).$$

Both values of n are possible; for there are two progressions which satisfy the given conditions, one having 6 terms, the other 10; these progressions are

$$15,\quad 13,\quad 11,\quad 9,\quad 7,\quad 5,$$

and $$15,\quad 13,\quad 11,\quad 9,\quad 7,\quad 5,\quad 3,\quad 1,\quad -1,\quad -3.$$

Another Solution.—Substituting the given values in the group

$$\left\{\begin{array}{l} l = a + (n - 1)\,d \\ s = \frac{n}{2}(a + l) \end{array}\right\},$$

we obtain the group

$$\left\{\begin{array}{r} l = 15 - 2(n - 1) \\ 60 = \frac{n}{2}(15 + l) \end{array}\right\};$$

whence, $\quad l = 5$ or -3, and $n = 6$ or 10.

3. Given $a = 275$, $l = 5$, and $n = 46$, to find d and s.

Ans. $\left\{\begin{array}{l} d = -6, \\ s = 6440. \end{array}\right.$

4. Given $d = 5$, $n = 8$, and $s = 156$, to find a and l.

Ans. $\left\{\begin{array}{l} a = 2, \\ l = 37. \end{array}\right.$

5. Form an A. P. of 6 terms whose extremes shall be 7 and 37.

Ans. 7, 13, 19, 25, 31, 37.

6. Given $a = 3$, $n = 60$, and $s = 372$, to find d and l.

Ans. $\begin{cases} d = 2, \\ l = 121. \end{cases}$

7. What is the sum of the terms of an A. P. formed by inserting 9 arithmetical means between 9 and 109? *Ans.* 649.

8. Find the sum of the first n terms of the progression 1, 2, 3, 4, 5, 6, ...

Ans. $s = \frac{n}{2}(1 + n)$.

9. Find the sum of the first n terms of the progression 1, 3, 5, 7, 9, ... *Ans.* $s = n^2$.

10. Sum to 30 terms the progression 116, 108, 100, ...

Ans. $s = 0$.

11. Sum to n terms the progression 9, 11, 13, 15, ...

Ans. $s = n(8 + n)$.

12. Are the squares of $x^2 - 2x - 1$, $x^2 + 1$, and $x^2 + 2x - 1$ in A. P.?

13. A sets out from a place and travels 1 mile the first day, 2 the second, 3 the third, and so on. Five days later B sets out from the same place and travels 12 miles a day in the same direction as A. How long will A travel before he is overtaken by B?

Ans. 8 or 15 days.

14. A sets out from a place and travels 1 mile the first day, 2 the second, 3 the third, and so on. B sets out a days later from the same place and travels b miles a day in the same direction as A. How long will A travel before he is overtaken by B?

Ans. $\dfrac{2b - 1 \pm \sqrt{(2b-1)^2 - 8ab}}{2}$ days.

Show that B will *never* overtake A if $a > \dfrac{(2b-1)^2}{8b}$.

15. A sets out from a place and travels 1 mile the first day, 3 the second, 5 the third, and so on. B sets out three days later from the same place and in the same direction as A, and travels 12 miles the first day, 13 the second, 14 the third, and so on. How long will A travel before he is overtaken by B?

Ans. 5 or 14 days.

16. The distance from P to Q is 165 miles. A sets out from P toward Q and travels 1 mile the first day, 2 the second, 3 the third, and so on. At the same time B sets out from Q toward P, and travels 20 miles the first day, 18 the second, 16 the third, and so on. When will they meet?

Ans. At the end of 10 or 33 days.

Do A and B meet twice?

ARITHMETICAL MEAN.

508. ***The Arithmetical Mean*** of two or more quantities is the quotient obtained by dividing their sum by their number. Thus, the arithmetical mean of a and b is $\frac{a+b}{2}$, and the arithmetical mean of 1, 7, 11, and 5 is 6.

509. To find the arithmetical mean of the terms of an A. P.

Denoting the arithmetical mean by M, we have, by definition,

$$M = \frac{s}{n}.$$

But
$$s = \frac{n}{2}(a + l);$$

$$\therefore \quad M = \frac{a + l}{2}.$$

510. To find a and l when M, d, and n are given.

$$\left\{ \begin{aligned} a &= \frac{s}{n} - \frac{(n-1)d}{2} \\ l &= \frac{s}{n} + \frac{(n-1)d}{2} \end{aligned} \right\} \text{ (\textbf{506}, 8).}$$

Substituting M for $\frac{s}{n}$, we have

$$\left\{ \begin{aligned} a &= M - \frac{(n-1)d}{2} \\ l &= M + \frac{(n-1)d}{2} \end{aligned} \right\}.$$

511. *PROBLEMS.*

1. Find five numbers in A. P. whose sum is 25, and whose continued product is 945.

Denote the arithmetical mean by M and the common difference by x; then

$$M = \frac{25}{5} = 5;$$

and

$$\left.\begin{cases} \text{the first} & \text{term} = 5 - 2x \\ \text{the second} & \text{term} = 5 - x \\ \text{the third} & \text{term} = 5 \\ \text{the fourth} & \text{term} = 5 + x \\ \text{the fifth} & \text{term} = 5 + 2x \end{cases}\right\} \textbf{(510)};$$

$$\therefore \quad (5-2x)(5-x)5(5+x)(5+2x) = 3125 - 625x^2 + 20x^4$$
$$= 945;$$

whence, $\qquad x = \pm 2 \text{ or } \pm\frac{1}{2}\sqrt{218}.$

The required numbers are therefore 1, 3, 5, 7, 9,

or $\quad 5-\sqrt{218}, \quad 5-\frac{1}{2}\sqrt{218}, \quad 5, \quad 5+\frac{1}{2}\sqrt{218}, \quad 5+\sqrt{218}.$

2. Find four numbers in A. P. whose sum is 32, and the sum of whose squares is 276. *Ans.* 5, 7, 9, 11.

3. Find three numbers in A. P., the sum of whose squares is 1232, and the square of whose arithmetical mean exceeds the product of the extremes by 16. *Ans.* 16, 20, 24.

4. Find four numbers in A. P. whose sum is 28, and whose continued product is 585. *Ans.* 1, 5, 9, 13.

5. The sum of the squares of the first and last of four numbers in A. P. is 200, and the sum of the squares of the second and third is 136: find the numbers. *Ans.* 2, 6, 10, 14.

6. Find the first term and the common difference in an A. P. of eighteen terms, in which the sum of any two terms equidistant from the extremes is $31\frac{1}{2}$, and the product of the extremes is $85\frac{1}{2}$.

Ans. $\begin{cases} a = 3, \\ d = 1\frac{1}{2}. \end{cases}$

GEOMETRICAL PROGRESSION.

512. ***A Geometrical Progression,*** or a ***Progression by Quotient,*** is a series in which the quotient obtained by dividing the second term by the first is equal to the quotient obtained by dividing any other term by the preceding one. Thus, 1, 3, 9, 27, 81 is a geometrical progression.

A geometrical progression is sometimes called a ***Geometrical Series.***

For brevity we shall sometimes use G. P. for the phrase *geometrical progression.*

513. ***The Extremes*** of a G. P. are the first term and the last term; the other terms are the ***Means.***

514. ***The Ratio*** of a G. P. is the quotient obtained by dividing any term by the one which precedes it. Thus, in the progression 1, 3, 9, 27, 81 the ratio is 3.

515. ***An Increasing G. P.*** is one in which the ratio is greater than 1. Thus, 1, 2, 4, 8, 16 is an increasing G. P.

516. ***A Decreasing G. P.*** is one in which the ratio is less than 1. Thus, 64, 16, 4, 1, $\frac{1}{4}$ is a decreasing G. P.

517. ***An Infinite Decreasing G. P.*** is one in which the ratio is less than 1, and the number of terms is infinite.

518. ***Notation.*** In treating geometrical progressions we shall use the following notation:

$a =$ the first term of the progression,
$l =$ the last or n^{th} term,
$r =$ the ratio,
$n =$ the number of terms,
$s =$ the sum of all the terms.

Thus, in the G. P. 1, 3, 9, 27, 81,

$$a = 1, \quad l = 81, \quad r = 3, \quad n = 5, \quad s = 121.$$

519. To find l when a, r, and n are given.

The first term is a, the second term is ar, the third term is ar^2,

the fourth term is ar^3, and so on; hence the n^{th} term is ar^{n-1}; that is,

$$l = ar^{n-1} = \frac{a}{r} \cdot r^n.$$

COR.—If $n = \infty$ and $r < 1$; then $l = \frac{a}{r} \times 0 = 0$; that is, *The last term of an infinite decreasing G. P. is* 0.

520. To find s when a, l, and r are given.

$$s = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} \quad . \quad . \quad . \quad (1).$$

Multiplying (1) by r,

$$rs = ar + ar^2 + ar^3 + ar^4 + \dots + ar^n \quad . \quad . \quad . \quad (2).$$

Subtracting (1) from (2),

$$rs - s = ar^n - a \quad . \quad . \quad . \quad (3);$$

whence,
$$s = \frac{ar^n - a}{r - 1} \quad . \quad . \quad . \quad (4).$$

Substituting lr for ar^n (**519**), (4) becomes,

$$s = \frac{lr - a}{r - 1} \quad . \quad . \quad . \quad (5).$$

COR.—If $n = \infty$ and $r < 1$; then $l = 0$ (**519**, COR.), and (5) becomes
$$s = \frac{a}{1 - r} \quad . \quad . \quad . \quad (6);$$
that is,

The sum of the terms of an infinite decreasing G. P. is equal to the quotient obtained by dividing the first term by 1 *minus the ratio.*

521. *In a G. P. the product of any two terms equidistant from the extremes is equal to the product of the extremes.*

Let x denote a term which has m terms before it, and y a term which has m terms after it; then

$$\left.\begin{cases} x = ar^m \\ y = l\left(\frac{1}{r}\right)^m \end{cases}\right\} (\mathbf{519});$$

whence,
$$xy = al.$$

522. To insert any number of geometrical means between two given quantities.

Let a and b be the given quantities, and let it be required to insert m geometrical means between them; that is, let it be required to form a G. P. whose extremes are a and b and the number of whose terms is $m + 2$.

Substituting b for l and $m + 2$ for n in the formula of Art. **519**, we have

$$b = ar^{m+1};$$

whence, $$r = \sqrt[m+1]{\frac{b}{a}}.$$

By multiplying a by the ratio we obtain the second term; by multiplying the second term by the ratio we obtain the third term; and so on.

Example.—Insert three geometrical means between 7 and 112.

$$r = \sqrt[4]{\frac{112}{7}} = 2;$$

hence the required progression is 7, 14, 28, 56, 112.

523. To find the continued product of the terms of a G. P.

Denoting the required product by P, we have

$$P = a \times ar \times ar^2 \times ar^3 \times \ldots . l \quad . \quad . \quad . \quad (1).$$

Inverting the order of the factors in the second member of (1), we have

$$P = l \times \ldots . ar^3 \times ar^2 \times ar \times a \quad . \quad . \quad . \quad (2).$$

Multiplying (1) by (2),

$$P^2 = (al)(al)(al) \ldots . (al) = (al)^n \ldots . (3) \ (\mathbf{521});$$

whence, $$P = \sqrt{(al)^n} \quad . \quad . \quad . \quad (4).$$

524. To find any two of the quantities a, l, r, n, and s when the three others are given.

The group $$\left\{ \begin{array}{l} l = ar^{n-1} \\ s = \dfrac{lr - a}{r - 1} \end{array} \right\}$$

contains the five quantities a, l, r, n, and s; hence any two of them may be found when the three others are given.

The ten cases are given in the following table as an exercise for the student. The value of n in the last four cases cannot be found without a knowledge of the properties of logarithms. This part of the work must therefore be deferred until Chapter XXI shall have been read.

NO.	GIVEN	TO FIND	RESULTING FORMULÆ.
1	a, r, n	l, s	$l = ar^{n-1}$, $s = \dfrac{ar^n - a}{r-1}$.
2	l, r, n	a, s	$a = \dfrac{l}{r^{n-1}}$, $s = \dfrac{l\,(r^n - 1)}{r^n - r^{n-1}}$.
3	n, r, s	a, l	$a = \dfrac{s\,(r-1)}{r^n - 1}$, $l = \dfrac{(r-1)\,sr^{n-1}}{r^n - 1}$.
4	a, l, n	r, s	$r = \left(\dfrac{l}{a}\right)^{\frac{1}{n-1}}$, $s = \dfrac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$.
5	a, n, s	r, l	$ar^n - rs = a - s$, $l\,(s-l)^{n-1} = a(s-a)^{n-1}$.
6	l, n, s	a, r	$a\,(s-a)^{n-1} = l\,(s-l)^{n-1}$, $(s-l)r^n - sr^{n-1} = -l$.
7	a, r, l	s, n	$s = \dfrac{lr - a}{r-1}$, $n = \dfrac{\log. l - \log. a}{\log. r} + 1$.
8	a, l, s	r, n	$r = \dfrac{s-a}{s-l}$, $n = \dfrac{\log. l - \log. a}{\log.(s-a) - \log.(s-l)} + 1$.
9	a, r, s	l, n	$\begin{cases} l = \dfrac{a + s\,(r-1)}{r}, \\ n = \dfrac{\log.\,[a + s\,(r-1)] - \log. a}{\log. r}. \end{cases}$
10	l, r, s	a, n	$\begin{cases} a = lr - s\,(r-1), \\ n = \dfrac{\log. l - \log.\,[lr - s\,(r-1)]}{\log. r} + 1. \end{cases}$

525. *PROBLEMS.*

1. The first term of a G. P. is 3, the ratio is 2, and the number of terms is 12. Find the last term and the sum of the terms.

We have given $a = 3,\ r = 2,\ n = 12$;

$$\therefore \qquad l = 3 \times 2^{11} = 6144,$$

and $$s = \frac{3 \times 2^{12} - 3}{2 - 1} = 12285 \ \ (\mathbf{524},\ 1).$$

2. Given $s = 1820,\ n = 6,$ and $r = 3$, to find a and l.

$$a = \frac{1820\,(3-1)}{3^6 - 1} = 5,$$

and $$l = \frac{1820\,(3-1)\,3^5}{3^6 - 1} = 1215 \ \ (\mathbf{524},\ 3).$$

After finding the value of a, we might have found the value of l from the formula $l = ar^{n-1}$.

3. Find the sum of an infinite decreasing G. P. of which the first term is 1, and the ratio $\frac{1}{2}$.

$$s = \frac{1}{1 - \frac{1}{2}} = 2 \ \ (\mathbf{520},\ \text{Cor.}).$$

4. Given $a = 1,\ l = 512,$ and $s = 1023$, to find r.

Ans. $r = 2$.

5. Insert two geometrical means between 24 and 192.

Ans. 24, 48, 96, 192.

6. Multiply $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots$ to infinity by $\frac{1}{5} - \frac{1}{25} + \frac{1}{125} - \frac{1}{625} + \ldots$ to infinity.

Ans. $\frac{2}{3}$.

7. Find the value of x in the equation

$$1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \text{ to infinity} = 2.$$

Ans. $x = \frac{1}{2}$.

8. Find the ratio of an infinite decreasing G. P. of which the first term is 1, and the sum of the terms $\frac{5}{4}$. *Ans.* $r = \frac{1}{5}$.

9. Find the first term of an infinite decreasing G. P. of which the ratio is $\frac{1}{m}$, and the sum of the terms $\frac{m}{m-1}$. *Ans.* $a = 1$.

10. Find the sum of the first n terms of the G. P. whose m^{th} term is $(-1)^m a^{4m}$.

$$Ans.\ s = \frac{a^4}{a^4+1}[(-1)^n a^{4n} - 1].$$

11. Find the ratio of an infinite decreasing G. P., in which each term is ten times the sum of all the terms which follow it.

Ans. $r = \frac{1}{11}$.

12. Find the sum of the first n terms of a G. P. whose first term is a, and third term c.

$$Ans.\ s = \frac{\sqrt{\frac{c^n}{a^{n-2}}} - a}{\sqrt{\frac{c}{a}} - 1}.$$

GEOMETRICAL MEAN.

526. ***The Geometrical Mean*** of n quantities is the n^{th} root of their product. Thus, the geometrical mean of a and b is $\sqrt{ab}$, and the geometrical mean of 1, 3, 6, and 72 is 6.

527. To find the geometrical mean of the terms of a G. P.

Denoting the geometrical mean by M, and the product of the terms by P, we have, by definition,

$$M = \sqrt[n]{P}.$$

But $$P = \sqrt{(al)^n}\ \mathbf{(523)};$$

$$\therefore\quad M = \sqrt{al}.$$

528. To find a and l when M, r, and n are given.

$$\left\{\begin{array}{l} a = \dfrac{l}{r^{n-1}} = \dfrac{al}{ar^{n-1}} \\ \\ l = ar^{n-1} = \dfrac{alr^{n-1}}{l} \end{array}\right\} \quad \textbf{(519).}$$

Substituting M^2 for al, we have

$$\left\{\begin{array}{l} a = \dfrac{M^2}{ar^{n-1}} \\ \\ l = \dfrac{M^2 r^{n-1}}{l} \end{array}\right\};$$

whence,
$$\left\{\begin{array}{l} a = \dfrac{M}{\sqrt{r^{n-1}}} \\ \\ l = M\sqrt{r^{n-1}} \end{array}\right\}.$$

COR. 1.—If $M = \sqrt{xy}$ and $r = x^{-1}y$, all the terms of the progression are of the first degree and rational if n is *even*.

The sum of the exponents in the ratio $x^{-1}y$ is 0; hence all the terms are of the same degree. The ratio $x^{-1}y$ is rational; hence the terms are either all rational or all irrational. It is sufficient, therefore, to show that the first term is of the first degree and rational.

$$a = \frac{M}{\sqrt{r^{n-1}}} = \frac{\sqrt{xy}}{\sqrt{(x^{-1}y)^{n-1}}} = \sqrt{\frac{xy}{x^{1-n}y^{n-1}}} = \sqrt{x^n y^{2-n}} = x^{\frac{n}{2}} y^{\frac{2-n}{2}},$$

which is of the first degree and rational when n is even.

COR. 2.—If $M = xy$ and $r = x^{-1}y$, all the terms of the progression are of the second degree and rational if n is *odd*.

$$a = \frac{M}{\sqrt{r^{n-1}}} = \frac{xy}{\sqrt{(x^{-1}y)^{n-1}}} = \frac{\sqrt{x^2y^2}}{\sqrt{x^{1-n}y^{n-1}}} = \sqrt{\frac{x^2y^2}{x^{1-n}y^{n-1}}} = \sqrt{x^{n+1}y^{3-n}}$$

$$= x^{\frac{n+1}{2}} y^{\frac{3-n}{2}} = x^{\frac{n+1}{2}} y^{1-\frac{n-1}{2}},$$

which is of the second degree and rational when n is *odd*.

529. *PROBLEMS.*

1. Find the first term of a G. P. of three terms whose geometrical mean is xy and ratio $\frac{y}{x}$.

Substituting 3 for n, xy for M, and $\frac{y}{x}$ for r in the formula

$$a = \frac{M}{\sqrt{r^{n-1}}}, \qquad \text{we have} \qquad a = \frac{xy}{\sqrt{\left(\frac{y}{x}\right)^2}} = \frac{xy}{\frac{y}{x}} = x^2.$$

2. Write a G. P. of three terms whose geometrical mean is xy and ratio $\frac{y}{x}$. *Ans.* x^2, xy, y^2.

3. Write a G. P. of four terms whose geometrical mean is $\sqrt{xy}$ and ratio $\frac{y}{x}$. *Ans.* $\frac{x^2}{y}, x, y, \frac{y^2}{x}$.

4. Write a G. P. of five terms whose geometrical mean is xy and ratio $\frac{y}{x}$. *Ans.* $\frac{x^3}{y}, x^2, xy, y^2, \frac{y^3}{x}$.

5. Write a G. P. of six terms whose geometrical mean is $\sqrt{xy}$ and ratio $\frac{y}{x}$. *Ans.* $\frac{x^3}{y^2}, \frac{x^2}{y}, x, y, \frac{y^2}{x}, \frac{y^3}{x^2}$.

6. The sum of three numbers in G. P. is 26, and the sum of their squares is 364. What are the numbers?

Denote the geometrical mean by xy and the ratio by $\frac{y}{x}$; then by the conditions of the problem,

$$x^2 + xy + y^2 = 26 \quad . \quad . \quad . \quad (1),$$

and

$$x^4 + x^2y^2 + y^4 = 364 \quad . \quad . \quad . \quad (2).$$

Transposing xy in (1) and squaring the result,

$$x^4 + 2x^2y^2 + y^4 = 676 - 52xy + x^2y^2 \quad . \quad . \quad . \quad (3);$$

whence,

$$x^4 + x^2y^2 + y^4 = 676 - 52xy \quad . \quad . \quad . \quad (4).$$

Since the first members of (2) and (4) are identical,

$$364 = 676 - 52xy \quad . \quad . \quad . \quad (5);$$

whence,
$$y = \frac{6}{x} \quad . \quad . \quad . \quad (6).$$

Substituting this value of y in (1) and solving the resulting equation, we find $\quad x^2 = 18 \text{ or } 2.$

Squaring (6) and substituting for x^2 its value, we find

$$y^2 = 2 \text{ or } 18.$$

From (6) $\quad xy = 6.$

Hence the numbers are 18, 6, and 2.

7. The sum of four numbers in G. P. is 15, and the sum of their squares is 85. What are the numbers?

Denote the geometrical mean by $\sqrt{xy}$ and the ratio by $\frac{y}{x}$; then by the conditions of the problem,

$$\frac{x^2}{y} + x + y + \frac{y^2}{x} = 15 \quad . \quad . \quad . \quad (1),$$

and
$$\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = 85 \quad . \quad . \quad . \quad (2).$$

Assume $x + y = z$, and $xy = p$; then

$$x^2 + y^2 = z^2 - 2p, \quad \text{and} \quad x^3 + y^3 = z^3 - 3zp.$$

Substituting z for $x + y$ in (1) and $z^2 - 2p$ for $x^2 + y^2$ in (2),

$$\frac{x^2}{y} + z + \frac{y^2}{x} = 15 \quad . \quad . \quad . \quad (3),$$

and
$$\frac{x^4}{y^2} + z^2 - 2p + \frac{y^4}{x^2} = 85 \quad . \quad . \quad . \quad (4).$$

Transposing z in (3) and $z^2 - 2p$ in (4),

$$\frac{x^2}{y} + \frac{y^2}{x} = 15 - z \quad . \quad . \quad . \quad (5),$$

and
$$\frac{x^4}{y^2} + \frac{y^4}{x^2} = 85 - z^2 + 2p \quad . \quad . \quad . \quad (6).$$

Squaring (5) and transposing $2xy$ or $2p$,

$$\frac{x^4}{y^2} + \frac{y^4}{x^2} = (15 - z)^2 - 2p \quad . \quad . \quad . \quad (7).$$

Since the first members of (6) and (7) are identical,

$$(15 - z)^2 - 2p = 85 - z^2 + 2p \quad . \quad . \quad . \quad (8);$$

whence, $$2z^2 - 30z - 4p = -140 \quad . \quad . \quad . \quad (9).$$

Clearing (5) of fractions,

$$x^3 + y^3 = (15 - z)\,xy = 15p - pz \quad . \quad . \quad . \quad (10).$$

Substituting $z^3 - 3zp$ for $x^3 + y^3$, (10) becomes

$$z^3 - 3zp = 15p - pz \quad . \quad . \quad . \quad (11);$$

whence, $$p = \frac{z^3}{15 + 2z} \quad . \quad . \quad . \quad (12).$$

Substituting this value of p in (9), clearing of fractions, transposing and reducing, we obtain

$$15z^2 + 85z = 1050 \quad . \quad . \quad . \quad (13);$$

whence, $$z = 6 \text{ or } -\frac{35}{3}.$$

Substituting 6 for z in (12), we find

$$p = 8.$$

We then have the equations

$$x + y = 6 \quad . \quad . \quad . \quad (14),$$

and $$xy = 8 \quad . \quad . \quad . \quad (15);$$

whence, $$x = 4 \text{ or } 2, \quad \text{and} \quad y = 2 \text{ or } 4.$$

The required numbers are therefore 1, 2, 4, 8.

The second value of z leads to imaginary results.

In the solution of such problems as this and the preceding one, the terms of the progression may be represented by

$$x,\ xy,\ xy^2,\ xy^3,\ xy^4,\ \ldots.$$

but the notation we have used is generally preferable.

8. The sum of three numbers in G. P. is 210, and the last exceeds the first by 90. What are the numbers?

Ans. 30, 60, 120.

9. The continued product of three numbers in G. P. is 216, and the sum of the squares of the extremes is 328. What are the numbers? *Ans.* 2, 6, 18.

10. The continued product of three numbers in G. P. is 64, and the sum of their cubes is 584. What are the numbers?

Ans. 2, 4, 8.

11. The sum of 120 dollars was divided among four persons in such a manner that the shares were in A. P. If the second and third had each received 12 dollars less, and the fourth 24 dollars more, the shares would have been in G. P. Find the shares.

Ans. \$3, \$21, \$39, \$57.

12. The sum of six numbers in G. P. is 189, and the sum of the third and fourth is 36. What are the numbers?

Ans. 3, 6, 12, 24, 48, 96.

TREATMENT OF SERIES BY THE DIFFERENTIAL METHOD.

530. ***The First Order of Differences*** of a series is the series obtained by subtracting each term of the given series from the following term; ***the Second Order of Differences*** is the series obtained by subtracting each term of the first order of differences from the following term; ***the Third Order of Differences*** is obtained from the second in the same way as the second is from the first; and so on. Thus,

If the given series be	1, 4, 9, 16, 25, ...
The 1st order of differences is	3, 5, 7, 9, ...
The 2d order of differences is	2, 2, 2, ...
The 3d order of differences is	0, 0, ...

531. ***The Differential Method*** is the process of finding any term of a series, or the sum of any number of its terms, by means of the *successive orders of differences.*

532. **To find the n^{th} term of a series.**

Let $a, b, c, d, e, \ldots$ be the proposed series.

The 1st order of differences is $b-a,\ c-b,\ d-c,\ e-d, \ldots$

The 2d order of differences is $c-2b+a,\ d-2c+b,\ e-2d+c, \ldots$

The 3d order of differences is $d-3c+3b-a,\ e-3d+3c-b, \ldots$

The 4th order of differences is $e-4d+6c-4b+a, \ldots$

. . . .

Denote the first term of the first order of differences by d_1, the first term of the second order of differences by d_2, the first term of the third order of differences by d_3, and so on; then

$$\left\{\begin{array}{l} d_1 = b - a \\ d_2 = c - 2b + a \\ d_3 = d - 3c + 3b - a \\ d_4 = e - 4d + 6c - 4b + a \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots \quad (1);$$

$$\text{whence,} \left\{\begin{array}{l} b = a + d_1 \\ c = a + 2d_1 + d_2 \\ d = a + 3d_1 + 3d_2 + d_3 \\ e = a + 4d_1 + 6d_2 + 4d_3 + d_4 \\ \ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots \quad (2).$$

The coefficients in the value of c, the *third* term of the proposed series, are 1, 2, 1, which are the coefficients of the expansion of $(x+a)^2$; the coefficients in the value of d, the *fourth* term, are 1, 3, 3, 1, which are the coefficients of the expansion of $(x+a)^3$; the coefficients in the value of e, the *fifth* term, are 1, 4, 6, 4, 1, which are the coefficients of the expansion of $(x+a)^4$; and so on. Hence the coefficients in the value of the n^{th} term are the coefficients of the expansion of $(x+a)^{n-1}$. Therefore, denoting the n^{th} term of the series by T_n,

$$T_n = a + (n-1)d_1 + \frac{(n-1)(n-2)}{\lfloor 2}d_2 + \frac{(n-1)(n-2)(n-3)}{\lfloor 3}d_3 + \ldots \quad (A).$$

EXAMPLES.

1. Find the 12th term of the series 1, 4, 9, 16, 25, ...
In this example $a = 1$, $d_1 = 3$, $d_2 = 2$, $d_3 = 0$, and $n = 12$. Substituting these values in (A), we obtain

$$T_{12} = 1 + 11 \times 3 + \frac{11 \times 10 \times 2}{2} = 144.$$

2. Find the 9th term of the series 1, 4, 8, 13, 19, ...
Ans. 53.

3. Find the 15th term of the series 1, 4, 10, 20, 35, ...
Ans. 680.

4. Find the 8th term of the series 1, 6, 21, 56, 126, ...
Ans. 771.

5. Find the 20th term of the series 1, 8, 27, 64, 125, ...
Ans. 8000.

6. Find the n^{th} term of the series 1, 3, 6, 10, 15, 21, ...
Ans. $\frac{n(n+1)}{2}$.

Is $n(n+1)$ divisible by 2? Why?

7. Find the n^{th} term of the series 1, 4, 10, 20, 35, ...
Ans. $\frac{n(n+1)(n+2)}{6}$.

Is $n(n+1)(n+2)$ divisible by 6? Why?

8. Find the n^{th} term of the series 1, 5, 15, 35, 70, 126, ...
Ans. $\frac{n(n+1)(n+2)(n+3)}{24}$.

Is $n(n+1)(n+2)(n+3)$ divisible by 24? Why?

533. To find the sum of n terms of a series.

Let $\qquad a, b, c, d, e, \quad . \quad . \quad . \quad (1)$

be the proposed series, and denote the sum of n terms of it by S_n. Let us assume the series

$$0, a, a + b, a + b + c, a + b + c + d, \quad . \quad . \quad . \quad (2).$$

Now it is evident that the sum of n terms of (1) is equal to the $(n+1)^{th}$ term of (2).

Denoting the $(n+1)^{th}$ term of (2) by T_{n+1}, the first term of the first order of differences of (2) by D_1, the first term of the second order of differences by D_2, and so on, we have by (A),

$$T_{n+1}=0+nD_1+\frac{n(n-1)}{\lfloor 2}D_2+\frac{n(n-1)(n-2)}{\lfloor 3}D_3+\dots \quad (3).$$

But $T_{n+1}=S_n$, $D_1=a$, $D_2=b-a=d_1$, $D_3=c-2b+a=d_2$, and so on. Hence, by substitution, (3) becomes

$$S_n=na+\frac{n(n-1)}{\lfloor 2}d_1+\frac{n(n-1)(n-2)}{\lfloor 3}d_2+\dots \quad (B).$$

EXAMPLES.

1. Find the sum of 10 terms of the series 1, 4, 9, 16, 25, ...
In this example $a=1$, $d_1=3$, $d_2=2$, $d_3=0$, and $n=10$.
Substituting these values in (B), we obtain

$$S_{10}=10+\frac{10\times 9\times 3}{2}+\frac{10\times 9\times 8\times 2}{6}=385.$$

2. Find the sum of 20 terms of the series 1, 3, 6, 10, 15, 21, ...
Ans. 1540.

3. Find the sum of 12 terms of the series 1, 5, 14, 30, 55, 91, ...
Ans. 2366.

4. Find the sum of 10 terms of the series 1, 4, 13, 37, 85, 166, ...
Ans. 2755.

5. Find the sum of n terms of the series 1, 3, 6, 10, 15, 21, ...
Ans. $\frac{n(n+1)(n+2)}{6}$.

INTERPOLATION.

534. ***Interpolation*** is the process of inserting between two consecutive terms of a given series a term or terms which shall conform to the law of that series.

535. ***The Formula for Interpolation*** is that given for finding the n^{th} term of a series by the differential method.

EXAMPLES.

Given $\left\{\begin{array}{l}\sqrt[3]{21} = 2.758924 \\ \sqrt[3]{22} = 2.802039 \\ \sqrt[3]{23} = 2.843867 \\ \sqrt[3]{24} = 2.884499 \\ \sqrt[3]{25} = 2.924018\end{array}\right\}$ to find the cube root of any intermediate number by the differential method.

1. Find the cube root of 21.75.

The operation of finding the orders of differences may be conveniently arranged as follows:

NO.	CUBE ROOTS.	d_1	d_2	d_3	d_4
21	2.758924				
22	2.802039	+.043115			
23	2.843867	+.041828	—.001287		
24	2.884499	+.040632	—.001196	+.000091	
25	2.924018	+.039519	—.001113	+.000083	—.000008

The distance between any two consecutive terms of the given series is 1; hence the value of n which corresponds to the required term is $1\frac{3}{4}$; that is, the required term is $\frac{3}{4}$ of the way from the first to the second term. Substituting in (A) $1\frac{3}{4}$ for n, 2.758924 for a, .043115 for d_1, — .001287 for d_2, .000091 for d_3, — .000008 for d_4, and reducing, we find $T_{1\frac{3}{4}} = \sqrt[3]{21.75} = 2.791385$.

2. Find the cube root of 21.325. *Ans.* 2.773083.

3. Find the cube root of 21.875. *Ans.* 2.796722.

4. Find the cube root of 21.4568. *Ans.* 2.778785.

5. Find the cube root of 22.4. *Ans.* 2.812613.

6. Find the cube root of 22.684. *Ans.* 2.830783.

7. Find the cube root of $22\frac{3}{4}$. *Ans.* 2.833525.

DEVELOPMENT OF EXPRESSIONS INTO SERIES.

536. ***To Develop*** or ***Expand*** an expression is to convert it into a series.

537. To develop a fraction into a series by division.

EXAMPLES.

1. Convert $\dfrac{1}{1-x}$ into an infinite series.

$$\begin{array}{l|l}
1 & 1-x \\ \hline
\underline{1-x} & 1+x+x^2+x^3+\text{etc.} \\
\quad\quad x & \\
\quad\quad \underline{x-x^2} & \\
\quad\quad\quad\quad x^2 & \\
\quad\quad\quad\quad \underline{x^2-x^3} & \\
\quad\quad\quad\quad\quad\quad x^3 &
\end{array}$$

$$\therefore \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \text{etc. to infinity} \quad . \; . \; . \quad (1).$$

If $x = \frac{1}{2}$, (1) becomes $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \text{etc.}$ . . . . (2).

If $x = 1$, (1) becomes $\infty = 1 + 1 + 1 + 1 + \text{etc.}$ . . . . (3).

If $x = 2$, (1) becomes $-1 = 1 + 2 + 4 + 8 + \text{etc.}$ . . . . (4).

How is this result to be explained?

Convert each of the following fractions into an infinite series:

2. $\dfrac{a}{a+x}$. *Ans.* $1 - \dfrac{x}{a} + \dfrac{x^2}{a^2} - \dfrac{x^3}{a^3} + \dfrac{x^4}{a^4} - \ldots.$

3. $\dfrac{a}{a-x}$. *Ans.* $1 + \dfrac{x}{a} + \dfrac{x^2}{a^2} + \dfrac{x^3}{a^3} + \dfrac{x^4}{a^4} + \ldots.$

4. $\dfrac{1+x}{1-x}$. *Ans.* $1 + 2x + 2x^2 + 2x^3 + 2x^4 + \ldots.$

5. $\dfrac{a+x}{a^2+x^2}$. *Ans.* $\dfrac{1}{a} - \dfrac{x^2}{a^3} + \dfrac{x^4}{a^5} - \dfrac{x^6}{a^7} + \dfrac{x^8}{a^9} - \ldots.$

6. $\dfrac{1}{1-a+a^2}$. *Ans.* $1 + a - a^3 - a^4 + a^6 + a^7 - a^9 - a^{10} + \ldots.$

538. **To develop an expression of the form of $\sqrt{m \pm n}$ by extracting the indicated root.**

EXAMPLES.

1. Convert $\sqrt{1+x}$ into an infinite series.

$$\begin{array}{r|l}
 & 1 + x\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \text{etc.}\right. \\
 & 1 \\
2 + \frac{x}{2}\Big) & \quad x \\
 & \quad x + \frac{x^2}{4} \\
2 + x - \frac{x^2}{8}\Big) & \qquad - \frac{x^2}{4} \\
 & \qquad - \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64} \\
2 + x - \frac{x^2}{4} + \frac{x^3}{16}\Big) & \qquad\qquad \frac{x^3}{8} - \frac{x^4}{64} \\
 & \qquad\qquad \frac{x^3}{8} + \frac{x^4}{16} - \frac{x^5}{64} + \frac{x^6}{256} \\
 & \qquad\qquad - \frac{5x^4}{64} + \frac{x^5}{64} - \frac{x^6}{256}
\end{array}$$

$$\therefore \qquad \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots\cdot$$

Convert each of the following expressions into an infinite series:

2. $\sqrt{a-x}$.

$$\textit{Ans. } a^{\frac{1}{2}}\left(1 - \frac{x}{2a} - \frac{x^2}{2\cdot 4a^2} - \frac{3x^3}{2\cdot 4\cdot 6a^3} - \frac{3\cdot 5x^4}{2\cdot 4\cdot 6\cdot 8a^4} - \cdots\cdot\right).$$

3. $\sqrt{a^2+b^2}$.

$$\textit{Ans. } a + \frac{b^2}{2a} - \frac{b^4}{2\cdot 4a^3} + \frac{3b^6}{2\cdot 4\cdot 6a^5} - \frac{3\cdot 5b^8}{2\cdot 4\cdot 6\cdot 8a^7} + \cdots\cdot$$

4. $\sqrt{1-x}$.

$$\textit{Ans. } 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \cdots\cdot$$

539. To develop an expression by means of undetermined coefficients.

EXAMPLES.

1. Convert $\dfrac{1+2x}{1-3x}$ into an infinite series.

Assume $\dfrac{1+2x}{1-3x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots$ (1).

Multiplying (1) by $1-3x$, we obtain

$$1+2x = A + \begin{array}{r|}B\\-3A\end{array}\,x + \begin{array}{r|}C\\-3B\end{array}\,x^2 + \begin{array}{r|}D\\-3C\end{array}\,x^3 + \begin{array}{r|}E\\-3D\end{array}\,x^4 + \ldots \quad (2).$$

Equating the coefficients of like powers of x in the two members of (2),

$$\left\{\begin{array}{r}A = 1\\ B-3A = 2\\ C-3B = 0\\ D-3C = 0\\ E-3D = 0\\ \ldots\ldots\end{array}\right\};$$

whence, $A = 1$, $B = 5$, $C = 15$, $D = 45$, $E = 135$, ...

Substituting these values in (1),

$$\frac{1+2x}{1-3x} = 1 + 5x + 15x^2 + 45x^3 + 135x^4 + \ldots.$$

2. Convert $\dfrac{1}{3x-x^2}$ into an infinite series.

Assume $\dfrac{1}{3x-x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots$ (1).

Multiplying (1) by $3x-x^2$, we obtain

$$1 = 3Ax + \begin{array}{r|}3B\\-A\end{array}\,x^2 + \begin{array}{r|}3C\\-B\end{array}\,x^3 + \begin{array}{r|}3D\\-C\end{array}\,x^4 + \ldots \quad (2);$$

whence, $1 = 0$, which is absurd; hence the second member of (1) is not of the proper form.

Assume $\frac{1}{3x - x^2} = Ax^{-1} + Bx^0 + Cx + Dx^2 + Ex^3 + \ldots$ (3).

Multiplying (3) by $3x - x^2$,

$$1 = 3A + \left.\begin{matrix}3B \\ -A\end{matrix}\right| x + \left.\begin{matrix}3C \\ -B\end{matrix}\right| x^2 + \left.\begin{matrix}3D \\ -C\end{matrix}\right| x^3 + \ldots \quad (4).$$

Equating the coefficients of like powers of x in the two members of (4),

$$\left\{\begin{aligned} 3A &= 1 \\ 3B - A &= 0 \\ 3C - B &= 0 \\ 3D - C &= 0 \\ \ldots&\ldots \end{aligned}\right\};$$

whence, $A = \frac{1}{3}$, $B = \frac{1}{9}$, $C = \frac{1}{27}$, $D = \frac{1}{81}$, ...

Substituting these values in (3),

$$\frac{1}{3x - x^2} = \frac{x^{-1}}{3} + \frac{x^0}{9} + \frac{x}{27} + \frac{x^2}{81} + \ldots$$

$$= \frac{1}{3x} + \frac{1}{9} + \frac{x}{27} + \frac{x^2}{81} + \ldots$$

The proper form of the second member of the assumed identity may be determined in each case by observing what the given expression becomes when the *variable* is supposed to be zero. If the given expression becomes a finite quantity, the first term of the series will not contain the variable; if it becomes zero, the first term of the series will contain the variable; and if it becomes infinity the first term of the series will be of the form Ax^{-n}.

Convert each of the following expressions into an infinite series:

3. $\frac{1 - 2x}{1 - 3x}$. *Ans.* $1 + x + 3x^2 + 9x^3 + 27x^4 + 81x^5 + \ldots$

4. $\frac{1 + 2x}{1 - x - x^2}$. *Ans.* $1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 + \ldots$

5. $\frac{1 - x}{1 - 3x - 2x^2}$ *Ans.* $1 + 2x + 8x^2 + 28x^3 + 100x^4 + 356x^5 + \ldots$

6. $\dfrac{x(1+5x)}{(1-2x)^2}$. *Ans.* $x+9x^2+32x^3+92x^4+240x^5+\ldots$

7. $\dfrac{2}{3x-2x^2}$. *Ans.* $\dfrac{2}{3x}+\dfrac{4}{9}+\dfrac{8x}{27}+\dfrac{16x^2}{81}+\dfrac{32x^2}{243}+\ldots$

8. $\dfrac{1}{1+2x^2+3x^4}$.

Ans. $1-2x^2+x^4+4x^6-11x^8+10x^{10}+13x^{12}-\ldots$

RECURRING SERIES.

540. ***A Recurring Series*** is one which may be produced by expanding some rational fraction. Thus,

$$1+x+3x^2+9x^3+27x^4+81x^5+\ldots$$

is a recurring series, because it is the expansion of the fraction $\dfrac{1-2x}{1-3x}$ (**539**, 3). In this series all the terms after the first two *recur* according to a definite law.

541. ***The Generating Fraction*** of a recurring series is the fraction which can be converted into the given series. Thus, the generating fraction of the series $1+x+3x^2+9x^3+27x^4+\ldots$ is $\dfrac{1-2x}{1-3x}$.

542. In the series given in Art. **540**, each term after the second may be obtained by multiplying the preceding term by $3x$; and in the series $1+4x+11x^2+34x^3+101x^4+\ldots$ the sum of the products obtained by multiplying the first of any two consecutive terms by $3x^2$ and the second by $2x$ is equal to the next succeeding term. The expression by means of which any term of a series may be found when the preceding terms are known is called the ***Scale of Relation***. Thus, the scale of relation of the series $1+x+3x^2+9x^3+\ldots$ is $3x$, and the scale of relation of the series $1+4x+11x^2+34x^3+\ldots$ is $3x^2+2x$.

543. A recurring series is said to be of the n^{th} *order* when the number of terms in its scale of relation is n. Thus, the series $1+4x+11x^2+34x^3+\ldots$ is of the second order.

544. To find the scale of relation in a recurring series.

Let $a + b + c + d + e + \ldots$ be the proposed series.

1st. Suppose that the series is of the first order.

Let m denote the scale of relation; then $b = ma$; whence,

$$m = \frac{b}{a}.$$

2d. Suppose the series to be of the second order.

Let $m + n$ denote the scale of relation; then

$$\left\{\begin{array}{l} c = ma + nb \\ d = mb + nc \end{array}\right\} \textbf{(542-543)};$$

whence, $$m = \frac{c^2 - bd}{ac - b^2}, \quad \text{and} \quad n = \frac{ad - bc}{ac - b^2}.$$

3d. Suppose the series to be of the third order.

Let $m + n + r$ denote the scale of relation; then

$$\left\{\begin{array}{l} d = ma + nb + rc \\ e = mb + nc + rd \\ f = mc + nd + re \end{array}\right\}.$$

From this group of equations m, n, and r may be found.

If the series is of the fourth order, fifth order, sixth order, &c., the scale of relation may be found in a similar manner.

Cor. 1.—If the proposed series is of the n^{th} order, $n + 1$ consecutive terms must be given to enable us to find the scale of relation.

Cor. 2.—If we assume any proposed series to be of a higher order than it really is, one or more of the terms of the scale of relation will be found to be equal to zero.

If we assume any proposed series to be of a lower order than it really is, or if we attempt to find the scale of relation of a series which is not recurring, the error will appear if we attempt to apply the scale.

EXAMPLES.

Find the scale of relation in each of the following series:

1. $$1 + 4x + 10x^2 + 22x^3 + \ldots.$$

Assume the scale of relation to be $m + n$; then

$$\left\{\begin{array}{l} 10x^2 = m + 4nx \\ 22x^3 = 4mx + 10nx^2 \end{array}\right\};$$

whence, $m = -2x^2$, and $n = 3x$. Therefore the scale of relation is $-2x^2 + 3x$.

2. $1+6x+12x^2+48x^3+120x^4+\ldots.$ *Ans.* $6x^2+x$.

3. $1+2x+3x^2+4x^3+5x^4+\ldots.$ *Ans.* $-x^2+2x$.

4. $1+2x+8x^2+28x^3+100x^4+\ldots.$ *Ans.* $2x^2+3x$.

5. $1+x+5x^2+13x^3+41x^4+\ldots.$ *Ans.* $3x^2+2x$.

545. To find the generating fraction of a recurring series.

Let $a + b + c + d + e + \ldots.$ be the proposed series.
1st. Suppose the series to be of the first order.
Let m denote the scale of relation; then

$$\left\{\begin{array}{l} b = ma \\ c = mb \\ d = mc \\ e = md \\ \ldots\ldots \end{array}\right\};$$

whence, $b+c+d+e+\ldots. = m(a+b+c+d+e+\ldots.)$.

Hence, denoting the generating fraction or the sum of the series by F_1, we have

$$F_1 - a = mF_1;$$

whence, $$F_1 = \frac{a}{1-m} \quad \ldots \quad (O_1).$$

2d. Suppose the series to be of the second order.

Let $m + n$ denote the scale of relation; then

$$\left\{\begin{array}{l} c = ma + nb \\ d = mb + nc \\ e = mc + nd \\ f = md + ne \\ \dots\dots\dots \end{array}\right\};$$

whence,

$$c+d+e+f+\ldots = m(a+b+c+d+e+f+\ldots)+n(b+c+d+e+f+\ldots).$$

Hence, denoting the generating fraction by F_2, we have

$$F_2 - (a+b) = mF_2 + n(F_2 - a);$$

whence, $$F_2 = \frac{a+b-an}{1-m-n} \quad . \quad . \quad . \quad (O_2).$$

3d. Suppose the series to be of the third order.

Let $m + n + r$ denote the scale of relation; then

$$\left\{\begin{array}{l} d = ma + nb + rc \\ e = mb + nc + rd \\ f = mc + nd + re \\ g = md + ne + rf \\ \dots\dots\dots\dots\dots \end{array}\right\};$$

whence, $$d+e+f+g+\ldots = m(a+b+c+d+e+f+g+\ldots)$$
$$+n(b+c+d+e+f+g+\ldots)+r(c+d+e+f+g+\ldots).$$

Hence, denoting the generating fraction by F_3, we have

$$F_3 - (a+b+c) = mF_3 + n(F_3 - a) + r[F_3 - (a+b)];$$

whence,

$$F_3 = \frac{a+b+c-an-(a+b)r}{1-m-n-r} \quad . \quad . \quad . \quad (O_3).$$

If the series is of the fourth order, fifth order, sixth order, etc., the generating fraction may be found in a similar manner.

Sch.—The formulæ (O_1), (O_2), (O_3), etc., have been obtained on the hypothesis that the given series is infinite and converging.

EXAMPLES.

Find the generating fraction of each of the following series:

1. $1+4x+10x^2+22x^3+46x^4+\ldots.$

In this series the scale of relation is $-2x^2+3x$ (**544,** 1); hence (O_2) is applicable. Substituting $-2x^2$ for m, $3x$ for n, 1 for a, and $4x$ for b, we have

$$F_2=\frac{1+4x-3x}{1+2x^2-3x}=\frac{1+x}{1+2x^2-3x}.$$

2. $1+3x+4x^2+7x^3+11x^4+\ldots.$ *Ans.* $\frac{1+2x}{1-x-x^2}$.

3. $1+6x+12x^2+48x^3+120x^4+\ldots.$ *Ans.* $\frac{1+5x}{1-x-6x^2}$.

4. $1+2x-5x^2+26x^3-119x^4+\ldots.$ *Ans.* $\frac{1+6x}{1+4x-3x^2}$.

5. $1+4x+3x^3-2x^3+4x^4+17x^5+3x^6+\ldots.$

Ans. $\frac{1+3x+x^2}{1-x+2x^2-3x^3}$.

6. $1+3x+5x^2+7x^3+9x^4+\ldots.$ *Ans.* $\frac{1+x}{(1-x)^2}$.

REVERSION OF SERIES.

546. ***To Revert a Series*** containing an unknown quantity is to express the value of that unknown quantity in terms of the sum of the given series. Thus, to revert the series in the second member of the equation

$$y=ax+bx^2+cx^3+dx^4+ex^5+\ldots.$$

is to find the value of x in terms of y.

EXAMPLES.

1. Revert the series in the equation $y=x+x^2+x^3+x^4+\ldots.$

This is a recurring series whose generating fraction is $\frac{x}{1-x}$;

hence $y=\dfrac{x}{1-x}$; whence, $x=\dfrac{y}{1+y}=y-y^2+y^3-y^4+\ldots\ldots$

2. Revert the series in the equation $y=x-\dfrac{x^2}{2}+\dfrac{x^3}{4}-\dfrac{x^4}{8}+\ldots\ldots$

This is a recurring series whose generating fraction is $\dfrac{2x}{2+x}$;

hence $y=\dfrac{2x}{2+x}$; whence, $x=\dfrac{2y}{2-y}=y+\dfrac{y^2}{2}+\dfrac{y^3}{4}+\dfrac{y^4}{8}+\ldots\ldots$

A recurring series cannot be reverted by this method when the equation, formed by placing y, the sum of the given series, equal to the generating fraction, cannot be solved. The method used in the following example is applicable to any series.

3. Revert the series in the equation

$$y=ax+bx^2+cx^3+dx^4+\ldots\ldots\ldots \quad (1).$$

Assume $\quad x=\mathrm{A}y+\mathrm{B}y^2+\mathrm{C}y^3+\mathrm{D}y^4+\ldots\ldots\ldots \quad (2),$

in which the coefficients A, B, C, D, ... are undetermined.

Substituting this value of x in (1), we have

$$y=a\mathrm{A}y+\left.\begin{array}{r}b\mathrm{A}\\+a^2\mathrm{B}\end{array}\right|y^2+\left.\begin{array}{r}c\mathrm{A}\\+2ab\mathrm{B}\\+a^3\mathrm{C}\end{array}\right|y^3+\left.\begin{array}{r}d\mathrm{A}\\+b^2\mathrm{B}\\+2ac\mathrm{B}\\+3a^2b\mathrm{C}\\+a^4\mathrm{D}\end{array}\right|y^4+\ldots\ldots\ldots \quad (3).$$

$$\therefore \quad \left\{\begin{array}{l}a\mathrm{A}=1\\ b\mathrm{A}+a^2\mathrm{B}=0\\ c\mathrm{A}+2ab\mathrm{B}+a^3\mathrm{C}=0\\ d\mathrm{A}+b^2\mathrm{B}+2ac\mathrm{B}+3a^2b\mathrm{C}+a^4\mathrm{D}=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\};$$

whence,

$$\mathrm{A}=\frac{1}{a},\ \mathrm{B}=-\frac{b}{a^3},\ \mathrm{C}=\frac{2b^2-ac}{a^5},\ \mathrm{D}=-\frac{a^2d-5abc+5b^3}{a^7},\ \ldots\ldots$$

Substituting these values in (2),

$$x=\frac{1}{a}y-\frac{b}{a^3}y^2+\frac{2b^2-ac}{a^5}y^3-\frac{a^2d-5abc+5b^3}{a^7}y^4+\ldots\ldots\ldots \quad (4).$$

4. Revert the series in the equation

$$y = x + 2x^2 + 4x^3 + 8x^4 + \ldots .$$

In this series

$$a = 1, \quad b = 2, \quad c = 4, \; d = 8, \ldots .$$

Substituting these values in (4) of the preceding example, we have

$$x = y - 2y^2 + 4y^3 - 8y^4 + \ldots .$$

5. Revert the series in the equation

$$\frac{1}{4} = 2x - \frac{4x^2}{3} + \frac{6x^3}{5} - \frac{8x^4}{7} + \ldots .$$

and find the value of x.

Substituting $\frac{1}{4}$ for y, 2 for a, $-\frac{4}{3}$ for b, $\frac{6}{5}$ for c, $-\frac{8}{7}$ for d,.... in (4) of example 3, we have

$$x = \frac{\frac{1}{4}}{2} + \frac{\left(\frac{1}{4}\right)^2}{6} + \frac{13\left(\frac{1}{4}\right)^3}{360} + \frac{5\left(\frac{1}{4}\right)^4}{1512} + \ldots = .125 + .010416 + .000564$$
$$+ .000013 + \ldots = .135993 + .$$

6. Revert the series in the equation

$$y = x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \ldots .$$

Ans. $x = y - 3y^2 + 13y^3 - 67y^4 + 381y^5 - \ldots .$

7. Revert the series in the equation

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots .$$

Ans. $x = y + \frac{y^2}{\lfloor 2} + \frac{y^3}{\lfloor 3} + \frac{y^4}{\lfloor 4} + \frac{y^5}{\lfloor 5} + \ldots .$

8. Revert the series in the equation

$$y = x + x^3 + x^5 + x^7 + x^9 + \ldots .$$

Ans. $x = y - y^3 + 2y^5 - 5y^7 + 14y^9 - \ldots .$

9. Find the value of x in the equation

$$\tfrac{2}{5} = 5x - 20x^2 + 80x^3 - 320x^4 + 1280x^5 - \ldots .$$

Ans. $x = .117647 + .$

THE BINOMIAL FORMULA FOR ANY EXPONENT.

547. It has been shown that when n is a positive integer

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{\lfloor 2}a^2x^{n-2} + \ldots .$$

We now proceed to show that this formula is true, whether n is *positive* or *negative, entire* or *fractional.*

548. LEM.—*The value of* $\dfrac{x^n - y^n}{x-y}$, when $y = x$, is nx^{n-1}, *whether* n is *positive* or *negative, entire* or *fractional.*

1. When n is a positive integer.

The proposition has been shown to be true for this case (**461,** COR. 2).

2. When n is a positive fraction.

Let $n = \dfrac{p}{q}$, in which p and q are supposed to be positive integers. We are to show that $\left\{\dfrac{x^{\frac{p}{q}} - y^{\frac{p}{q}}}{x-y}\right\}_{y=x} = \dfrac{p}{q}x^{\frac{p}{q}-1}$.

$$\frac{x^{\frac{p}{q}} - y^{\frac{p}{q}}}{x-y} = \frac{\left(x^{\frac{1}{q}}\right)^p - \left(y^{\frac{1}{q}}\right)^p}{\left(x^{\frac{1}{q}}\right)^q - \left(y^{\frac{1}{q}}\right)^q} = \frac{\dfrac{\left(x^{\frac{1}{q}}\right)^p - \left(y^{\frac{1}{q}}\right)^p}{x^{\frac{1}{q}} - y^{\frac{1}{q}}}}{\dfrac{\left(x^{\frac{1}{q}}\right)^q - \left(y^{\frac{1}{q}}\right)^q}{x^{\frac{1}{q}} - y^{\frac{1}{q}}}}.$$

But, by the first case, $\left\{\dfrac{\left(x^{\frac{1}{q}}\right)^p - \left(y^{\frac{1}{q}}\right)^p}{x^{\frac{1}{q}} - y^{\frac{1}{q}}}\right\}_{y=x} = p\left(x^{\frac{1}{q}}\right)^{p-1}$,

and $\left\{\dfrac{\left(x^{\frac{1}{q}}\right)^q - \left(y^{\frac{1}{q}}\right)^q}{x^{\frac{1}{q}} - y^{\frac{1}{q}}}\right\}_{y=x} = q\left(x^{\frac{1}{q}}\right)^{q-1}$;

$$\therefore \quad \left(\frac{x^{\frac{p}{q}} - y^{\frac{p}{q}}}{x-y}\right)_{y=x} = \frac{p\left(x^{\frac{1}{q}}\right)^{p-1}}{q\left(x^{\frac{1}{q}}\right)^{q-1}} = \frac{p}{q}x^{\frac{p}{q}-1}.$$

3. When n is a negative integer.

Suppose m to be a positive integer and that $n = -m$. We are to show that $\left\{\dfrac{x^{-m} - y^{-m}}{x - y}\right\}_{y=x} = -mx^{-m-1}$.

$$\frac{x^{-m} - y^{-m}}{x - y} = -x^{-m}y^{-m}\left(\frac{x^m - y^m}{x - y}\right).$$

But, by the first case, $\left\{\dfrac{x^m - y^m}{x - y}\right\}_{y=x} = mx^{m-1}$.

$$\therefore \quad \left(\frac{x^{-m} - y^{-m}}{x - y}\right)_{y=x} = -x^{-m}x^{-m} \times mx^{m-1} = -mx^{-m-1}.$$

4. When n is a negative fraction.

Let $n = -\dfrac{p}{q}$, in which p and q are supposed to be positive integers. We are to show that $\left\{\dfrac{x^{-\frac{p}{q}} - y^{-\frac{p}{q}}}{x - y}\right\}_{y=x} = -\dfrac{p}{q}x^{-\frac{p}{q}-1}$.

$$\frac{x^{-\frac{p}{q}} - y^{-\frac{p}{q}}}{x - y} = -x^{-\frac{p}{q}}y^{-\frac{p}{q}}\left(\frac{x^{\frac{p}{q}} - y^{\frac{p}{q}}}{x - y}\right).$$

But, by the second case, $\left\{\dfrac{x^{\frac{p}{q}} - y^{\frac{p}{q}}}{x - y}\right\}_{y=x} = \dfrac{p}{q}x^{\frac{p}{q}-1}$.

$$\therefore \quad \left(\frac{x^{-\frac{p}{q}} - y^{-\frac{p}{q}}}{x - y}\right)_{y=x} = -x^{-\frac{p}{q}}x^{-\frac{p}{q}} \times \frac{p}{q}x^{\frac{p}{q}-1} = -\frac{p}{q}x^{-\frac{p}{q}-1}.$$

549. Let us now find the expansion of $(x + a)^n$, when n is positive or negative, entire or fractional.

$$x + a = x\left(1 + \frac{a}{x}\right); \quad \text{therefore} \quad (x + a)^n = x^n\left(1 + \frac{a}{x}\right)^n.$$

Hence, the expansion of $(x + a)^n$ may be obtained by multiplying that of $\left(1 + \dfrac{a}{x}\right)^n$ by x^n.

Put $z = \dfrac{a}{x}$; then $\left(1 + \dfrac{a}{x}\right)^n = (1 + z)^n$.

Assume $(1+z)^n = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots . \quad (1),$

in which A, B, C, D, E, ... are undetermined coefficients independent of z.

Suppose $z = 0$; then from (1), we have $A = 1$.

Substituting 1 for A in (1), we have

$$(1+z)^n = 1 + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots \quad . \quad . \quad . \quad (2).$$

Since (2) is to be true for all values of z, we may substitute any letter or any expression for z. Substituting u for z in (2), we have

$$(1+u)^n = 1 + Bu + Cu^2 + Du^3 + Eu^4 + \ldots \quad . \quad . \quad . \quad (3).$$

Subtracting (3) from (2), and dividing the result by the identity $(1+z) - (1+u) = z - u$,

$$\frac{(1+z)^n - (1+u)^n}{(1+z) \;\; - (1+u)} = B + C\left(\frac{z^2 - u^2}{z - u}\right) + D\left(\frac{z^3 - u^3}{z - u}\right) + \ldots \quad . \quad . \quad . \quad (4).$$

Now suppose $u = z$; then, by the Lemma, (4) becomes

$$n(1+z)^{n-1} = B + 2Cz + 3Dz^2 + 4Ez^3 + \ldots \quad . \quad . \quad . \quad (5).$$

Multiplying (5) by $1 + z$,

$$n(1+z)^n = \begin{array}{l|l|l|l} B+2C & z+3D & z^2+4E & z^3+\ldots \quad . \quad . \quad . \quad (6). \\ \quad\;\; + B & \;+2C & \;\;+3D & \end{array}$$

Multiplying (2) by n,

$$n(1+z)^n = n + nBz + nCz^2 + nDz^3 + nEz^4 + \ldots \quad . \quad . \quad . \quad (7).$$

Equating the second members of (6) and (7),

$$\begin{array}{l|l|l|l} B+2C & z+3D & z^2+4E & z^3+\ldots = n + nBz + nCz^2 + nDz^3 + \ldots (8). \\ \;\; + B & \;+2C & \;\;+3D & \end{array}$$

$$\therefore \qquad \left\{ \begin{array}{r} B = n \\ 2C + \;\; B = nB \\ 3D + 2C = nC \\ 4E + 3D = nD \\ \ldots\ldots\ldots\ldots \end{array} \right\};$$

whence,
$$\left\{\begin{array}{l} B = n \\ C = \dfrac{n(n-1)}{\lfloor 2} \\ D = \dfrac{n(n-1)(n-2)}{\lfloor 3} \\ E = \dfrac{n(n-1)(n-2)(n-3)}{\lfloor 4} \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}.$$

Substituting these values in (2), we have

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{\lfloor 2} z^2 + \frac{n(n-1)(n-2)}{\lfloor 3} z^3 + \ldots \quad (9).$$

Substituting for z its value, $\frac{a}{x}$, (9) becomes

$$\left(1+\frac{a}{x}\right)^n = 1 + n\frac{a}{x} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{a^2}{x^2} + \frac{n(n-1)(n-2)}{\lfloor 3} \cdot \frac{a^3}{x^3} + \ldots \quad (10).$$

Multiplying (10) by x^n,

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{\lfloor 2} a^2 x^{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3} a^3 x^{n-3} + \ldots \quad (11).$$

Cor.—If n is not a positive integer, the expansion of $(x + a)^n$ is an infinite series; for no one of the factors in the coefficients can be equal to zero under this hypothesis.

EXAMPLES.

Expand each of the following expressions:

1. $\dfrac{1}{a+b}$.

$$\frac{1}{a+b} = (a+b)^{-1} = a^{-1} + (-1)ba^{-2} + \frac{(-1)(-2)}{\lfloor 2} b^2 a^{-3} + \ldots$$
$$= \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \ldots$$

2. $\sqrt{1+a}$. *Ans.* $1 + \frac{1}{2}a - \frac{1}{8}a^2 + \frac{1}{16}a^3 + \ldots$

3. $(a-x)^{\frac{1}{2}}$. *Ans.* $a^{\frac{1}{2}}\left(1-\frac{x}{2a}-\frac{x^2}{2\cdot 4a^2}-\frac{3x^3}{2\cdot 4\cdot 6a^3}-\ldots\right)$.

4. $(1-x)^{\frac{1}{3}}$. *Ans.* $1-\frac{x}{3}-\frac{2x^2}{3\cdot 6}-\frac{2\cdot 5x^3}{3\cdot 6\cdot 9}-\ldots$.

5. $(a+b)^{\frac{1}{3}}$. *Ans.* $a^{\frac{1}{3}}\left(1+\frac{b}{3a}-\frac{2b^2}{3\cdot 6a^2}+\frac{2\cdot 5b^3}{3\cdot 6\cdot 9a^3}-\ldots\right)$.

6. $\frac{1}{a-b}$. *Ans.* $\frac{1}{a}+\frac{b}{a^2}+\frac{b^2}{a^3}+\frac{b^3}{a^4}+\frac{b^4}{a^5}+\ldots$.

7. $\frac{a}{(1-x)^2}$. *Ans.* $a(1+2x+3x^2+4x^3+5x^4+\ldots)$.

8. $(a-c^2)^{\frac{2}{3}}$. *Ans.* $a^{\frac{2}{3}}\left(1-\frac{2c^2}{3a}-\frac{2c^4}{3\cdot 6a^2}-\frac{2\cdot 4c^6}{3\cdot 6\cdot 9a^3}-\ldots\right)$.

9. $(1-a)^{-3}$. *Ans.* $1+3a+6a^2+10a^3+15a^4+21a^5+\ldots$.

10. $\frac{x}{\sqrt[5]{1-x}}$. *Ans.* $x+\frac{x^2}{5}+\frac{6x^3}{2\cdot 5^2}+\frac{6\cdot 11x^4}{2\cdot 3\cdot 5^3}+\ldots$.

550. SYNOPSIS FOR REVIEW.

CHAP. XX. SERIES.

- GENERAL DEFINITIONS.
  - *Series.*
  - *Terms.*
  - *Finite Series.—Infinite Series.*
  - *Converging Series.—Diverg'g Series.*
- ARITHMETICAL PROGRESSION.
  - *Extremes.—Common difference.*
  - *Increasing A. P.—Decreasing A. P.*
  - *Notation.*
  - *To find l, when a, d, and n are given.*
  - *To find s, when a, l, and n are given.*
  - *Sum of two terms equidistant from extremes.*
  - *To insert any number of arithmetical means between two given quantities.*
  - *To find any two of the quantities a, l, n, d, s, when the three others are given.*
  - *Arith. Mean.*
    - To find arith. mean.
    - To find a and l when M, d, n, are given.

SYNOPSIS FOR REVIEW—*Continued.*

CHAP. XX. SERIES. *Continued.*

- GEOMETRICAL PROGRESSION.
  - *Extremes.—Ratio.*
  - *Increasing G.P.—Decreasing G. P. —Infinite Decreasing G. P.*
  - *Notation.*
  - *To find l, when a, r, and n are given. Cor.*
  - *To find s, when a, l, and r are given. Cor.*
  - *Product of two terms equidistant from extremes.*
  - *To insert any number of geometric means between two given quan.*
  - *To find the continued product of the terms of a G. P.*
  - *To find any two of the quantities a, l, n, r, and s, when the three others are given.*
  - *Geom. Mean.*
    - To find geometrical mean.
    - To find a and l when M, r, and n are given. Cor. 1, 2.
- TREATM'T OF SERIES BY THE DIFFERENTIAL METHOD.
  - *Orders of differences.*
  - *Differential method.*
  - *To find the n^{th} term of a series.*
  - *To find the sum of n terms of a series.*
  - *Interpolation.*—Formula for Interpolation.
- DEVELOPMENT OF EXPRESS'N INTO SERIES.
  - *Development of fractions by division.*
  - *Development of expressions of the form of $\sqrt{m \pm n}$ by extracting the indicated root.*
  - *Development of expressions by means of Undetermined Coefficients.*
- RECURRING SERIES . .
  - *Generating Fraction.*
  - *Scale of Relation.*
  - *Order of Recurring Series.*
  - *To find the Scale of Relation in a Recurring Series. Cor. 1, 2.*
  - *To find the Generating Fraction.* Sch.
- REVERSION OF SERIES.
- BINOMIAL FORMULA FOR ANY EXPONENT.
  - *Lemma.*
  - *Cor.*

CHAPTER XXI.

LOGARITHMS AND EXPONENTIAL EQUATIONS.

LOGARITHMS.

551. ***The Logarithm*** of a number is the exponent by which some fixed number must be affected in order to produce the given number. The fixed number is called the ***Base of the System.*** Thus, in the equation $a^x = n$, x is the logarithm of n to the base a.

For brevity, the expression $\log_a n$ is sometimes used to denote the logarithm of n to the base a. Thus, $x = \log_a n$ expresses the same relation as $a^x = n$.

552. Any number except $+1$ and -1 may be used as the base; hence there may be an infinite number of systems of logarithms. There are only two systems, however, in general use, namely: Briggs' system, the base of which is 10, and Napier's system, the base of which is $2.718+$.

Briggs' system of logarithms is used more than that of Napier, and is hence called the *common system.*

553. If in the equation $a^x = n$ we suppose n to represent a perfect power of a, then x will be some integer; but if n is not a perfect power of a, then x will be a *mixed number* or a *fraction.*

554. ***The Characteristic*** of a logarithm is the integral part of it, and the ***Mantissa*** is the fractional part. Thus, the characteristic of $\log_9 243$ is 2, and the mantissa is .5; for

$$9^{2.5} = 9^{\frac{5}{2}} = 3^5 = 243.$$

GENERAL PROPERTIES OF LOGARITHMS.

555. *In any system the logarithm of* 1 *is* 0.

For $a^x = 1$ when $x = 0$ (**84**, Cor.).

556. *In any system the logarithm of the base is* 1.

For $a^x = a$ when $x = 1$.

557. *In a system whose base is greater than* 1, *the logarithm of* 0 *is* $-\infty$.

For $a^{-\infty} = \dfrac{1}{a^\infty}$, and $\dfrac{1}{a^\infty} = 0$ when $a > 1$.

558. *In a system whose base is less than* 1, *the logarithm of* 0 *is* $+\infty$.

For $a^\infty = 0$ when $a < 1$.

559. *In a system whose base is positive, a negative quantity has no real logarithm.*

For, if a is positive, a^x is positive, whether x is positive or negative. Thus, $10^2 = 100$, and $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$.

560. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Let $x = \log_a m$, and $y = \log_a n$;

then $m = a^x$, and $n = a^y$;

whence, $mn = a^x a^y = a^{x+y}$.

Therefore, $\log_a mn = x + y$ (**551**) $= \log_a m + \log_a n$.

561. *The logarithm of a quotient is equal to the remainder obtained by subtracting the logarithm of the divisor from that of the dividend.*

Dividing $m = a^x$ by $n = a^y$,

$$\frac{m}{n} = a^{x-y};$$

hence $\log_a \dfrac{m}{n} = x - y = \log_a m - \log_a n$.

562. *The logarithm of any power of a number is equal to the product of the exponent of the power and the logarithm of the number.*

Raising both members of the equation $m = a^x$ to the r^{th} power,

$$m^r = a^{rx};$$

$$\therefore \qquad \log_a(m^r) = rx = r\log_a m.$$

563. *The logarithm of any root of a number is equal to the quotient obtained by dividing the logarithm of the number by the index of the root.*

Extracting the r^{th} root of both members of the equation $m = a^x$,

$$\sqrt[r]{m} = a^{\frac{x}{r}};$$

$$\therefore \qquad \log_a(\sqrt[r]{m}) = \frac{x}{r} = \frac{\log_a m}{r}.$$

564. *EXAMPLES.*

Prove each of the following statements:

1. $\log(abc) = \log a + \log b + \log c.$

2. $\log\left(\frac{abc}{d}\right) = \log a + \log b + \log c - \log d.$

3. $\log(a^2b^3c^4) = 2\log a + 3\log b + 4\log c.$

4. $\log\left(\frac{a^2b^3c^4}{d^5}\right) = 2\log a + 3\log b + 4\log c - 5\log d.$

5. $\log\sqrt{abc} = \frac{1}{2}(\log a + \log b + \log c).$

6. $\log\sqrt{a^2 - b^2} = \frac{1}{2}[\log(a+b) + \log(a-b)].$

THE COMMON SYSTEM.

565. To find the characteristic of a logarithm in the common system.

In this system,

$$\begin{array}{llll} \log 10^0 = \log 1 & = 0, & \log 10^{-1} = \log .1 & = -1, \\ \log 10^1 = \log 10 & = 1, & \log 10^{-2} = \log .01 & = -2, \\ \log 10^2 = \log 100 & = 2, & \log 10^{-3} = \log .001 & = -3, \\ \log 10^3 = \log 1000 & = 3, & \log 10^{-4} = \log .0001 & = -4, \\ \log 10^4 = \log 10000 & = 4, & \log 10^{-5} = \log .00001 & = -5, \\ \cdots & & \cdots & \end{array}$$

Hence, supposing n to be a positive integer,

1st. The logarithm of a number between 10^n and 10^{n+1} is greater than n and less than $n+1$; its characteristic, therefore, is n. Now, the number of figures in the integral part of a number between 10^n and 10^{n+1} is $n+1$. Hence,

The characteristic of the common logarithm of an integer, or of a number composed of an integer and a decimal fraction, is positive and one less than the number of figures in the integral part of that number. Thus, the characteristic of $\log_{10} 258.045$ is 2.

2d. The logarithm of a decimal fraction between 10^{-n} and $10^{-(n+1)}$, that is, between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, is some negative number between $-n$ and $-(n+1)$; hence, if we agree that the *mantissa shall in all cases be positive*, the characteristic will be $-(n+1)$. Now, the number of ciphers preceding the first significant figure in a decimal fraction between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$ is n. Hence,

The characteristic of the common logarithm of a decimal fraction is negative and numerically one greater than the number of ciphers preceding the first significant figure in that fraction. Thus, the characteristic of $\log_{10} .0546$ is -2.

566. *If the ratio of two numbers is any perfect power of* 10, *the mantissas of their logarithms in the common system will be the same.*

This follows from Art. **561.** Thus, denoting the mantissa of $\log_{10} 5468$ by m,

$$\log\ 5468 = 3 + m,$$

$$\log 546.8 = \log\left(\frac{5468}{10}\right) = \log 5468 - \log 10 = 3 + m - 1 = 2 + m,$$

$$\log 54.68 = \log\left(\frac{5468}{100}\right) = \log 5468 - \log 100 = 3 + m - 2 = 1 + m,$$

$$\log 5.468 = \log\left(\frac{5468}{1000}\right) = \log 5468 - \log 1000 = 3 + m - 3 = 0 + m,$$

$$\log .05468 = \log\left(\frac{5468}{100000}\right) = \log 5468 - \log 100000 = 3 + m - 5$$
$$= -2 + m.$$

COMPUTATION OF LOGARITHMS.

567. To express the logarithm of a number in terms of that number and the base of the system.

Let x be the logarithm of n to the base a; then

$$a^x = n \quad . \quad . \quad . \quad (1).$$

Assume $\qquad a = 1 + m, \quad$ and $\quad n = 1 + p;$

then (1) becomes

$$(1 + m)^x = 1 + p \quad . \quad . \quad . \quad (2);$$

whence, $\qquad (1 + m)^{xy} = (1 + p)^y \quad . \quad . \quad . \quad (3).$

Expanding both members of (3) by the Binomial Formula,

$$1 + xym + \frac{xy(xy-1)}{\underline{|2}} m^2 + \frac{xy(xy-1)(xy-2)}{\underline{|3}} m^3 + \ldots .$$

$$= 1 + yp + \frac{y(y-1)}{\underline{|2}} p^2 + \frac{y(y-1)(y-2)}{\underline{|3}} p^3 + \ldots . \quad \ldots (4).$$

Dropping 1 from both members of (4) and dividing the result by y,

$$xm + \frac{x(xy-1)}{\lfloor 2}m^2 + \frac{x(xy-1)(xy-2)}{\lfloor 3}m^3 + \dots$$

$$= p + \frac{(y-1)}{\lfloor 2}p^2 + \frac{(y-1)(y-2)}{\lfloor 3}p^3 + \dots \quad \dots \quad (5).$$

Making $y = 0$, (5) becomes

$$xm - x\frac{m^2}{2} + x\frac{m^3}{3} - x\frac{m^4}{4} + x\frac{m^5}{5} - \dots =$$

$$p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \dots \quad \dots \quad (6);$$

whence,

$$x = \frac{p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \dots}{m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \frac{m^5}{5} - \dots} \quad \dots \quad (7).$$

But $x = \log_a n = \log_a (1 + p)$; hence, if we put

$$M \doteq \frac{1}{m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \frac{m^5}{5} - \dots},$$

we have

$$x = \log_a (1 + p) = M\left(p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \dots\right) \dots \quad (L).$$

The second member of (L) consists of two factors, namely: the series within the parenthesis, which depends only upon the number, and the quantity M, which depends only upon the base of the system.

The factor M, which depends only upon the base, is called the *modulus* of the system.

The series in (L) is called the ***Logarithmic Series.***

568. To find the Base of Napier's System.

Baron Napier arbitrarily assumed the modulus of his system to be unity. Making $M = 1$, and denoting the Base of Napier's System by e, (L) becomes

$$x = \log_e (1 + p) = p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \dots \quad (1).$$

Reverting the series in the second member of (1), we obtain

$$p = x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \frac{x^6}{\underline{|6}} + \dots \quad (2).$$

But $$e^x = 1 + p;$$

$$\therefore \quad e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \frac{x^6}{6} + \dots \quad (3).$$

Making $x = 1$ in (3), we have

$$e = 2 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \frac{1}{\underline{|6}} + \dots \quad (4).$$

Summing the series in (4) to nine terms, we find

$$e = 2.718282.$$

569. *The logarithm of a number in any system is equal to the product obtained by multiplying the modulus of that system by the Napierian logarithm of the same number.*

For $$\log_a (1 + p) = \mathrm{M}\left(p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \dots\right) \ (\mathbf{567}),$$

and $$\log_e (1 + p) = p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \dots \ (\mathbf{568});$$

$$\therefore \quad \log_a (1 + p) = \mathrm{M} \log_e (1 + p).$$

COR.—If $$1 + p = a,$$

we have $$\log_a a = \mathrm{M} \log_e a;$$

but $$\log_a a = 1 \ (\mathbf{556});$$

$$\therefore \quad 1 = \mathrm{M} \log_e a;$$

whence, $$\mathrm{M} = \frac{1}{\log_e a}.$$

Hence,

The modulus of any system is equal to the reciprocal of the Napierian logarithm of the base of the system.

570. To transform the Logarithmic Series into a Converging Series.

The formula

$$\log_a(1+p) = M\left(p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \dots\right) \quad . \; . \; . \; (1)$$

cannot be used for the computation of logarithms when $p > 1$, because the series in its second member does not converge.

Substituting $-p$ for p in (1), we have

$$\log_a(1-p) = M\left(-p - \frac{p^2}{2} - \frac{p^3}{3} - \frac{p^4}{4} - \frac{p^5}{5} - \dots\right) \quad . \; . \; . \; (2).$$

Subtracting (2) from (1), observing that $\log_a(1+p) - \log_a(1-p) = \log_a\left(\frac{1+p}{1-p}\right)$ (**561**),

$$\log_a\left(\frac{1+p}{1-p}\right) = 2M\left(p + \frac{p^3}{3} + \frac{p^5}{5} + \frac{p^7}{7} + \dots\right) \quad . \; . \; . \; (3).$$

Assume $p = \frac{1}{2z+1}$; then $\frac{1+p}{1-p} = \frac{z+1}{z}$.

Substituting these values in (3),

$$\log_a\left(\frac{z+1}{z}\right) = \log_a(z+1) - \log_a z =$$

$$2M\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \dots\right) \quad . \; . \; . \; (4).$$

For Napier's System (4) becomes

$$\log_e(z+1) - \log_e z =$$

$$2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \dots\right) \quad . \; . \; . \; . \; (5);$$

whence, by transposition,

$$\log_e(z+1) =$$

$$\log_e z + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \dots\right) \; . \; . \; . \; (N).$$

571. To compute a Table of Napierian Logarithms.

$\log_e 0 = -\infty$ (**557**),

$\log_e 1 = 0$ (**555**),

$$\log_e 2 = \log_e 1 + 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \ldots\right)$$
$$= 0.693147 \ (\mathbf{570}),$$

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} + \frac{1}{7\cdot 5^7} + \ldots\right)$$
$$= 1.098612,$$

$$\log_e 4 = \log_e 2^2 = 2\log_e 2 = 1.386294,$$

$$\log_e 5 = \log_e 4 + 2\left(\frac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7} + \ldots\right)$$
$$= 1.609438,$$

$$\log_e 6 = \log_e 2 + \log_e 3 = 1.791759,$$

$$\log_e 7 = \log_e 6 + 2\left(\frac{1}{13} + \frac{1}{3\cdot 13^3} + \frac{1}{5\cdot 13^5} + \ldots\right) = 1.945910,$$

$$\log_e 8 = \log_e 2^3 = 3\log_e 2 = 2.079442,$$

$$\log_e 9 = \log_e 3^2 = 2\log_e 3 = 2.197225,$$

$$\log_e 10 = \log_e 2 + \log_e 5 = 2.302585,$$

. . . .

572. To find the modulus of the common system.

Denoting the modulus of the common system by M, we have

$$M = \frac{1}{\log_e 10} = \frac{1}{2.302585} = .434294+.$$

573. To compute a table of common logarithms.

If we multiply the Napierian logarithm of a number by the modulus of the common system, the product will be the common logarithm of the same number. Thus,

$$\log_{10} 5 = 1.609438 \times .434294 = 0.698970.$$

TABLE OF COMMON LOGARITHMS FROM 1 TO 100.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

574. *EXAMPLES.*

1. Find the product of 9 and 7 by means of logarithms.

$$\text{Log}\,(9 \times 7) = \log 9 + \log 7\ (\mathbf{560}) = 0.954243 + 0.845098 = 1.799341.$$

The number corresponding to this logarithm is 63 (**573**).

2. Divide 210 by 7 by means of logarithms.

$$\text{Log}\left(\frac{210}{7}\right)=\log 210-\log 7=2.322219-0.845098=1.477121$$
$$=\log 30.$$

3. Find the square of 9 by means of logarithms.

4. Find the fourth root of 625 by means of logarithms.

5. Find the logarithm of $33\frac{1}{3}$.

6. Find the logarithm of $6^2 \times 7^3 \times 8^4$.

EXPONENTIAL EQUATIONS.

575. ***An Exponential Equation*** is one in which the unknown quantity occurs as an exponent. Thus, $a^x = n$ is an exponential equation.

576. To solve the exponential equation $a^x = n$.

Taking the logarithm of each member of this equation, we have

$$x \log a = \log n \ (\mathbf{562});$$

whence,
$$x = \frac{\log n}{\log a}.$$

EXAMPLES.

Solve each of the following equations:

1. $3^x = 27$. *Ans.* $x = 3$.

2. $5^x = 100$. *Ans.* $x = 2.861$.

3. $2^{\frac{3}{x}} = 4$. *Ans.* 1.5.

4. $ab^x = n$. *Ans.* $x = \dfrac{\log n - \log a}{\log b}$.

5. $ab^{\frac{1}{x}} = n$. *Ans.* $x = \dfrac{\log b}{\log n - \log a}$.

6. $a^{2x} - 2pa^x = b$. *Ans.* $x = \dfrac{\log\left(p \pm \sqrt{b + p^2}\right)}{\log a}$.

577. SYNOPSIS FOR REVIEW.

CHAPTER XXI.

- LOGARITHMS.
  - Log. of a number.
  - Base of a system.
  - Characteristic.—Mantissa.
  - General Properties.
    - *Log.* 1.
    - *Log. Base.*
    - *When Base* $>$ 1.
    - *When Base* $<$ 1.
    - *When Base is positive.*
    - *Log. of a Product.*
    - *Log. of a Quotient.*
    - *Log. of a Power.*
    - *Log. of a Root.*
  - Common System.
    - *To find the characteristic.*
    - *Mantissas of log. of two numbers whose ratio is a perfect power of* 10.
  - Computation.
    - *To express log. of a number in terms of that number and the base of the system.*
    - *Modulus.—Logarithmic series.*
    - *To find the base of Napier's system.*
    - *Log. number = Modulus × Napier's Log. same number.*
    - *Modulus of any system = reciprocal of the Napierian log. of base of the system.*
    - *To transform log. series into converging series.*
    - *To compute table of Napierian log.*
    - *To compute table of common log.*
- EXPONENTIAL EQUATIONS.

CHAPTER XXII.

COMPOUND INTEREST AND ANNUITIES.

COMPOUND INTEREST.

578. To find the amount of p dollars at compound interest for n years at r per cent. per annum.

At the end of the 1st year the amount will be

$$p + pr = p(1 + r);$$

at the end of the 2d year the amount will be

$$p(1 + r) + p(1 + r)r = p(1 + r)^2;$$

at the end of the 3d year the amount will be

$$p(1 + r)^2 + p(1 + r)^2 r = p(1 + r)^3;$$

and so on. Hence, denoting the required amount by A,

$$\mathrm{A} = p(1 + r)^n \quad . \quad . \quad . \quad (1).$$

Any one of the four quantities, A, p, n, and r may be found from this equation when the three others are given. The computation is most readily performed by means of logarithms. Taking the logarithm of each member of (1),

$$\log \mathrm{A} = \log p + n \log(1 + r) \quad . \quad . \quad . \quad (2) \ (\mathbf{560\text{-}562});$$

whence, $\quad \log p = \log \mathrm{A} - n \log(1 + r) \quad . \quad . \quad . \quad (3),$

$$\log(1 + r) = \frac{\log \mathrm{A} - \log p}{n} \quad . \quad . \quad . \quad (4),$$

and $$n = \frac{\log \mathrm{A} - \log p}{\log(1 + r)} \quad . \quad . \quad . \quad (5).$$

Example.—How much will \$500 amount to in five years at 6 per cent. compound interest?

Given $\left\{\begin{array}{l}\log 1.06 \quad\;\; = 0.025306 \\ \log 669.10 = 2.825491\end{array}\right\}.$

Substituting 500 for p, .06 for r, and 5 for n in (2), we have

$$\begin{aligned}\log A &= \log 500 + 5 \log 1.06 \\ &= 2.698970 + 5 \times .025306 = 2.825500.\end{aligned}$$

Since $\log 669.10 = 2.825491$, it follows that $A = \$669.10$.

ANNUITIES.

579. ***An Annuity*** is a sum of money which is payable annually. The term is also applied to a sum of money payable at *any* equal intervals of time.

580. To find the amount of an annuity of a dollars for n years at r per cent. per annum, when the interest is compounded every year.

The first payment a becomes due at the end of the first year, and in $n-1$ years this will amount to $a(1+r)^{n-1}$ (**578**); the second payment a becomes due at the end of the second year, and in $n-2$ years this will amount to $a(1+r)^{n-2}$; the third payment will amount to $a(1+r)^{n-3}$ in $n-3$ years; and so on. Hence, denoting the amount of the annuity by A,

$$\begin{aligned}A = a(1+r)^{n-1} + a(1+r)^{n-2} + a(1+r)^{n-3} + \ldots . \\ + a(1+r) + a \quad . \quad . \quad . \quad (1).\end{aligned}$$

By reversing the order of the terms in the second member of (1),

$$A = a + a(1+r) + a(1+r)^2 + \ldots . + a(1+r)^{n-1} \quad . \quad . \quad . \quad (2);$$

$$\text{whence, } A = \frac{a(1+r)^n - a}{(1+r) - 1} = a \cdot \frac{(1+r)^n - 1}{r} \quad . \quad . \quad . \quad (3).$$

581. To find the present value of an annuity of a dollars for n years, at r per cent. per annum, the interest being compounded every year.

Denoting the present value of the annuity by P,

$$P(1+r)^n = a\frac{(1+r)^n-1}{r} \quad . \quad . \quad . \quad (1)\ (\mathbf{578\text{-}580});$$

whence,
$$P = \frac{a}{r}\cdot\frac{(1+r)^n-1}{(1+r)^n} \quad . \quad . \quad . \quad (2).$$

Cor.—If $n=\infty$, (2) becomes

$$P = \frac{a}{r}.$$

582. SYNOPSIS FOR REVIEW.

CHAP. XXII.
- COMPOUND INTEREST.
  - To find the amount of p dollars at compound interest for n years at r per cent. per annum.
- ANNUITIES.
  - To find the amount of an annuity of a dollars for n years at r per cent. per annum, when the interest is compounded every year.
  - To find the present value of an annuity of a dollars for n years at r per cent. per annum, when the interest is compounded every year. *Cor.*

CHAPTER XXIII.

THEORY OF EQUATIONS.

DEFINITIONS.

583. Every equation of the n^{th} degree containing only one unknown quantity may be written under the form of

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots . + Kx + L = 0.$$

This equation is called *the general equation of the n^{th} degree.* The term L, which is called the *absolute* or *independent term,* may be considered as the coefficient of x^0.

584. ***A Function*** of a quantity is an expression containing that quantity. Thus, $ax^2 + bx$ is a function of x.

For brevity we shall sometimes use the symbol $f(x)$ to denote a *function of x.*

If $f(x)$ is *entire* and *rational* with reference to x, it is called a *rational integral function of x.*

In the present Chapter, when $f(x)$ is used without modification, it is understood to denote a rational integral function of x.

585. Any quantity, which substituted for x in $f(x)$ causes $f(x)$ to vanish, is a ***Root*** of the equation $f(x) = 0$.

GENERAL PROPERTIES.

586. *If $f(x)$ vanishes when $x = r$, the function is divisible by $x - r$.*

Suppose $f(x)$ to be divided by $x - r$, and the operation continued until a remainder is obtained which is independent of x.

Denote the quotient by Q and the remainder, if there be one, by R; we then have the identity

$$f(x) = Q(x - r) + R.$$

By hypothesis, $f(x)$ vanishes when $x = r$; and since Q is a rational integral function of x, it cannot become infinite when $x = r$; hence $Q(x - r)$ vanishes when $x = r$. Therefore R vanishes when $x = r$. But R does not contain x; hence it vanishes without regard to the value of x.

587. *If $f(x)$ is divisible by $x - r$, then r is a root of the equation $f(x) = 0$.*

Let Q denote the quotient obtained by dividing $f(x)$ by $x - r$; we shall then have the identity

$$f(x) = Q(x - r).$$

Now $Q(x - r)$ vanishes when $x = r$; hence $f(x)$ vanishes when $x = r$. It therefore follows that r is a root of the equation $f(x) = 0$ (**585**).

588. *If the equation $f(x) = 0$ is of the n^{th} degree, it has n roots, and no more.*

Let a represent a root of the equation

$$f(x) = 0 \quad . \quad . \quad . \quad (1);$$

then $f(x)$ is divisible by $x - a$ (**586**). The quotient obtained by dividing $f(x)$ by $x - a$ will be of the $(n-1)^{th}$ degree. Denoting the quotient by $f_1(x)$, (1) may be written

$$(x - a) f_1(x) = 0 \quad . \quad . \quad . \quad (2).$$

Again, let b represent a root of the equation

$$f_1(x) = 0 \quad . \quad . \quad . \quad (3),$$

which is of the $(n-1)^{th}$ degree; then $f_1(x)$ is divisible by $x - b$. The quotient obtained by dividing $f_1(x)$ by $x - b$ will be of the $(n-2)^{th}$ degree. Denoting this quotient by $f_2(x)$, (2) may be written

$$(x - a)(x - b) f_2(x) = 0 \quad . \quad . \quad . \quad (4).$$

By continuing this process, $f(x)$ will ultimately be resolved into n binomial factors, $x-a, x-b, x-c, x-d, \ldots, (x-k), x-l$.

$$\therefore\ f(x) = (x-a)(x-b)(x-c)(x-d)\ldots(x-k)(x-l)\ldots(5).$$

Now $f(x)$ vanishes when x is equal to any one of the n quantities $a, b, c, d, \ldots k, l$; hence $f(x)=0$ has n roots. This equation has no more than n roots, for if we ascribe to x a value m which is not one of the n values $a, b, c, d, \ldots k, l$, the value of $f(x)$ becomes $(m-a)(m-b)(m-c)(m-d)\ldots(m-k)(m-l)$, which is not zero, because each factor is different from zero.

COR.—If a is a root of the equation $f(x)=0$, then $f(x)=(x-a)f_1(x)$, where $f_1(x)$ is one degree lower than $f(x)$; hence the remaining roots of the equation $f(x)=0$ can be found if we can solve the equation $f_1(x)=0$. In like manner, if a and b are roots of the equation $f(x)=0$, then $f(x)=(x-a)(x-b)f_2(x)$; hence the remaining roots of the equation $f(x)=0$ can be found if we can solve the equation $f_2(x)=0$, which is two degrees lower than the equation $f(x)=0$.

589. To find an equation when its roots are given.

Let $a, b, c, d, \ldots k$ be the n roots of an equation; then

$$(x-a)(x-b)(x-c)(x-d)\ldots(x-k)=0$$

will be the equation required; for each of the n quantities, $a, b, c, d, \ldots k$ is a root of this equation, and it has no other roots.

MULTIPLICATION BY DETACHED COEFFICIENTS.

590. To multiply a rational integral function of x by $x \pm a$, by means of detached coefficients.

EXAMPLES.

1. Multiply x^3+5x^2-6x+4 by $x-3$.

Since the coefficients of the product do not depend upon x, the product may be found as follows:

$1+5-\ \ 6+\ \ 4$	Detached coefficients of multiplicand.
$1-3$	" " " multiplier.
$1+5-\ \ 6+\ \ 4$	Detached coefficients of 1st partial product.
$-3-15+18-12$	" " " 2d " "
$1+2-21+22-12$	Detached coefficients of product.

Hence the product is $x^4+2x^3-21x^2+22x-12$.

Since the coefficients of the first partial product are identical with those of the multiplicand, this operation may be still further abridged as follows:

$$\begin{array}{r|l}
-3 & 1+5-\ \ 6+\ \ 4 \quad \text{Detached coefficients of multiplicand.} \\
 & \underline{\ \ -3-15+18-12} \\
 & 1+2-21+22-12 \quad \text{" " " product.}
\end{array}$$

Multiplying 1, the coefficient of the first term of the multiplicand, by -3, and adding the product to 5, we obtain 2; multiplying 5 by -3, and adding the product to -6, we obtain -21; multiplying -6 by -3, and adding the product to 4, we obtain 22; and multiplying 4 by -3, we obtain -12.

When multiplication is performed in this way, the terms should be arranged according to the powers of x; and if a term is wanting, its place should be filled with a cipher.

2. Multiply $x^4+6x^2+5x-10$ by $x-5$.

$$\begin{array}{r|l}
-5 & 1+0+6+\ \ 5-10 \\
 & \underline{\ \ -5-0-30-25+50} \\
 & 1-5+6-25-35+50
\end{array}$$

Hence the product is $x^5-5x^4+6x^3-25x^2-35x+50$.

3. Multiply $x^5-4x^3+6x^2-8x+15$ by $x+8$.

$$\begin{array}{r|l}
8 & 1+0-4+\ \ 6-\ \ 8+15 \\
 & \underline{\ \ +8+0-32+48-64+120} \\
 & 1+8-4-26+40-49+120
\end{array}$$

Product, $x^6+8x^5-4x^4-26x^3+40x^2-49x+120$.

4. Multiply x^7-4x^3+6x-7 by $x+3$.

Ans. $x^8+3x^7-4x^4-12x^3+6x^2+11x-21$.

DIVISION BY DETACHED COEFFICIENTS.

591. To divide a rational integral function of x by $x \pm a$, by means of detached coefficients.

EXAMPLES.

1. Divide $x^3 - 9x^2 + 26x - 24$ by $x - 4$.

Since the coefficients of the quotient do not depend upon x, the quotient may be found as follows:

$$\begin{array}{lrl}
\text{Coeffic'ts of dividend,} & 1-9+26-24 & \underline{|\,1-4} \text{ Coefficients of divisor.} \\
 & \underline{1-4\qquad\quad} & \;\;1-5+6 \quad \text{“} \qquad \text{“ quotient.} \\
 & -5+26\quad\; & \\
 & \underline{-5+20\qquad} & \\
 & 6-24 & \\
 & \underline{6-24} & \\
\end{array}$$

Hence the quotient is $x^2 - 5x + 6$.

This operation may be still further abridged as follows:

$$\begin{array}{r|l}
1 - 9 + 26 - 24 & -4 \quad . \quad . \quad . \quad (A). \\
-4 + 20 - 24 & \\
\hline
1 - 5 + \;\; 6 + \;\; 0 &
\end{array}$$

The coefficient of the first term of the quotient is evidently 1. Multiplying 1, the first coefficient in the dividend, by -4, and subtracting the product from -9, we obtain -5, which is the second coefficient in the quotient; multiplying -5 by -4, and subtracting the product from 26, we obtain 6, which is the third coefficient in the quotient; and multiplying 6 by -4, and subtracting the product from -24, we obtain 0.

We may substitute addition for subtraction in (A), if we multiply by $+4$; thus,

$$\begin{array}{r|l}
1-9+26-24 & 4 \quad . \quad . \quad . \quad (B). \\
+4-20+24 & \\
\hline
1-5+\;\;6+\;\;0 &
\end{array}$$

When division is performed by means of detached coefficients, the terms should be arranged according to the powers of x; and if a term is wanting, its place should be filled with a cipher.

The process used in (B) is called *Synthetic Division.*

2. Divide $x^4 - 3x^3 - 15x^2 + 49x - 12$ by $x - 5$.

$$\begin{array}{l|l} 1-3-15+49-\;12 & 5 \\ \;\;\;+5+10-25+120 & \\ \hline 1+2-\;5+24+108 & \end{array}$$

Hence the quotient is $x^3 + 2x^2 - 5x + 24$, and the remainder is 108.

3. Divide $x^4 - 8x^3 - 11x^2 + 198x - 360$ by $x - 7$.

$$\begin{array}{l|l} 1-8-11+198-360 & 7 \\ \;\;\;+7-\;\;7-126+504 & \\ \hline 1-1-18+\;\;72+144 & \end{array}$$

Hence the quotient is $x^3 - x^2 - 18x + 72$, and the remainder is 144.

4. Divide $x^3 + 5x^2 + 2x - 8$ by $x + 4$.

$$\begin{array}{l|l} 1+5+2-8 & -4 \\ \;\;\;-4-4+8 & \\ \hline 1+1-2+0 & \end{array}$$

Hence the quotient is $x^2 + x - 2$.

592. *GENERAL EXAMPLES.*

1. Show that 1 is a root of the equation

$$x^3 + 3x^2 - 16x + 12 = 0.$$

That the first member of this equation is divisible by $x - 1$ may be proved as follows:

$$\begin{array}{l|l} 1+3-16+12 & 1\ \textbf{(591)}; \\ \;\;\;+1+\;\;4-12 & \\ \hline 1+4-12+\;\;0 & \end{array}$$

hence 1 is a root (**587**).

2. Show that 3 is a root of the equation $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

3. Show that -7 is a root of the equation $x^4 + 2x^3 - 34x^2 + 12x + 35 = 0$.

4. Show that -1 and -2 are roots of the equation $x^5 - 4x^4 + 22x^2 - 25x - 42 = 0$.

5. Show that $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are roots of the equation $x^4 - 2x^3 + x^2 + 10x - 30 = 0$.

6. One root of the equation $x^3 + 5x^2 + 2x - 8 = 0$ is 1; what are the other roots?

$$\begin{array}{r|l} 1+5+2-8 & 1 \\ +1+6+8 & \\ \hline 1+6+8+0 & \end{array}$$

$x^2 + 6x + 8 = 0$ (**588**, Cor.); whence, $x = -2$ or -4.

7. Two roots of the equation $x^4 - 5x^3 - 7x^2 + 29x + 30 = 0$ are -1 and -2: what are the other roots?

$$\begin{array}{rrrrr|l} 1 & -5 & -7 & +29 & +30 & -1 \\ & -1 & +6 & +1 & -30 & \\ \hline 1 & -6 & -1 & +30 & +0 & -2 \\ & -2 & +16 & -30 & & \\ \hline 1 & -8 & +15 & +0 & & \end{array}$$

$x^2 - 8x + 15 = 0$; whence, $x = 3$ or 5.

8. Three roots of the equation $x^5 - 4x^4 + 22x^2 - 25x - 42 = 0$ are -1, -2, 3; what are the other roots?

Ans. $2 + \sqrt{-3}$, $2 - \sqrt{-3}$.

9. Two roots of the equation $x^4 - 3x^3 - 4x^2 + 30x - 36 = 0$ are 2 and -3; what are the other roots?

Ans. $2 + \sqrt{-2}$, $2 - \sqrt{-2}$.

10. One root of the equation $x^3 - 1 = 0$ is 1; what are the other roots?

Ans. $\frac{1}{2}(-1 \pm \sqrt{-3})$.

11. Find the equation whose roots are $1, \ -2, \ -4$.

The required equation is

$$(x-1)[x-(-2)][x-(-4)] = (x-1)(x+2)(x+4) = 0.$$

The indicated multiplication may be performed as follows:

$$\begin{array}{r|rrrr}
2 & 1 & -1 & & \\
 & & +2 & -2 & \\
\hline
4 & 1 & +1 & -2 & \\
 & & +4 & +4 & -8 \\
\hline
 & 1 & +5 & +2 & -8
\end{array} \qquad \textbf{(590)};$$

hence, $x^3 + 5x^2 + 2x - 8 = 0$ is the required equation in its simplest form.

12. Find the equation whose roots are $3, \ -2, \ -1, \ 5$.

Ans. $x^4 - 5x^3 - 7x^2 + 29x + 30 = 0$.

13. Find the equation whose roots are $1+\sqrt{-5}, \ 1-\sqrt{-5}, \ \sqrt{5}, \ -\sqrt{5}$.

Ans. $x^4 - 2x^3 + x^2 + 10x - 30 = 0$.

14. Find the equation whose roots are $-1, -2, 3, 2+\sqrt{-3}, 2-\sqrt{-3}$.

Ans. $x^5 - 4x^4 + 22x^2 - 25x - 42 = 0$.

15. Find the equation whose roots are a, b, c.

Ans. $x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = 0$.

16. Find the equation whose roots are a, b, c, d.

Ans. $x^4 - (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 - (abc+abd+acd+bcd)x + abcd = 0$.

593. To find the relation between the coefficients of $f(x)$ and the roots of the equation $f(x) = 0$.

Suppose the terms of $f(x)$ to be arranged according to the descending powers of x and that the coefficient of the first term is 1; then

1. *The coefficient of the second term with its sign changed is equal to the sum of the roots* (**592**, 15, 16);

2. *The coefficient of the third term is equal to the sum of the products of the roots, taken two and two;*

3. *The coefficient of the fourth term with its sign changed is equal to the sum of the products of the roots, taken three and three; and so on.*

4. *If the degree of the equation is even, the absolute term is equal to the product of all the roots. If the degree of the equation is odd, the absolute term with its sign changed is equal to the product of all the roots.*

By a method similar to that employed in Art. **472** it may be proved that these laws are true universally.

COR. 1.—If the roots of $f(x)=0$ are all negative, each term of $f(x)$ is positive.

COR. 2.—If the roots of $f(x)=0$ are all positive, the signs of the terms of $f(x)$ will be alternately $+$ and $-$.

COR. 3.—If the second term of $f(x)$ does not appear, the sum of the roots of the equation $f(x)=0$ is equal to zero. Thus, the sum of the roots of the equation $x^3-2x+4=0$ is zero.

COR. 4.—If $f(x)$ has no absolute term, at least one of the roots of $f(x)=0$ is zero. Thus, one root of the equation $x^3-2x^2+3x=0$ is 0.

COR. 5.—The absolute term of $f(x)$ is divisible by each root of the equation $f(x)=0$.

COR. 6.—Let $a, b, c, d, \ldots . l$ denote the roots of the equation $x^n+\mathrm{A}x^{n-1}+\mathrm{B}x^{n-2}+\ldots . +\mathrm{K}x+\mathrm{L}=0$; then

$$-\mathrm{A}=a+b+c+d+\ldots . +l,$$

and

$$\mathrm{B}=ab+ac+\ldots . +bd+be+\ldots . ;$$

whence,

$$\mathrm{A}^2-2\mathrm{B}=a^2+b^2+c^2+d^2+\ldots . +l^2;$$

that is, $\mathrm{A}^2-2\mathrm{B}$ is equal to the sum of the squares of the roots of the proposed equation. Hence, if $\mathrm{A}^2-2\mathrm{B}$ is *negative*, the roots of the equation cannot be all *real*. Thus, the roots of the equation $x^5-4x^4+22x^3-25x-42=0$ are not all real, for $(-4)^2-2\times 22$ is negative.

594. *An equation whose coefficients are integers, that of its first term being unity, cannot have a root which is a rational fraction.*

Let the equation be

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Kx + L = 0 \quad . \quad . \quad . \quad (1),$$

in which the coefficients A, B, ... K, L are supposed to be integers.

Suppose, if possible, that (1) has a rational fractional root which in its lowest terms is expressed by $\frac{a}{b}$. Substituting $\frac{a}{b}$ for x in (1), and multiplying the resulting equation by b^{n-1}, we obtain

$$\frac{a^n}{b} + Aa^{n-1} + Ba^{n-2}b + \ldots\ldots + Kab^{n-2} + Lb^{n-1} = 0 \ldots \quad (2);$$

whence,

$$\frac{a^n}{b} = -(Aa^{n-1} + Ba^{n-2}b + \ldots\ldots + Kab^{n-2} + Lb^{n-1}) \ldots \quad (3).$$

The second member of (3) is an *integer*, and its first member is an *irreducible fraction*. Hence $\frac{a}{b}$ cannot be a root of the proposed equation.

595. *If $a+b\sqrt{-1}$ is a root of an equation whose coefficients are real, then will $a-b\sqrt{-1}$ be a root of that equation.*

Let $a+b\sqrt{-1}$ be a root of the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Kx + L = 0 \quad . \quad . \quad . \quad (1),$$

in which the coefficients are supposed to be real, then will $a-b\sqrt{-1}$ be a root of that equation.

Since $a+b\sqrt{-1}$ is a root of (1),

$$(a+b\sqrt{-1})^n + A(a+b\sqrt{-1})^{n-1} + B(a+b\sqrt{-1})^{n-2} + \ldots\ldots$$
$$+ K(a+b\sqrt{-1}) + L = 0 \quad . \quad . \quad . \quad (2).$$

If we expand those terms of (2) which contain $a+b\sqrt{-1}$, the resulting equation will contain some terms which are *real* and

some which are *imaginary*. Since the coefficients A, B, C, ... and the *even* powers of $b\sqrt{-1}$ are real, it follows that $\sqrt{-1}$ will occur only in connection with the *odd* powers of b. Denoting the sum of the *real* terms by P, and the sum of the imaginary terms by $Q\sqrt{-1}$, we have

$$P + Q\sqrt{-1} = 0 \quad . \quad . \quad . \quad (3);$$

whence,
$$P = -Q\sqrt{-1} \quad . \quad . \quad . \quad (4).$$

To satisfy (4) we must have $P = 0$ and $Q = 0$, for a real quantity cannot be equal to an imaginary quantity.

Now if $a - b\sqrt{-1}$ be substituted for x in (1), its first member, when expanded, will differ from the result obtained by expanding the first member of (2) *only* in the sign of the odd powers of $b\sqrt{-1}$; that is, the first member of (1) may be represented by $P - Q\sqrt{-1}$ when $a - b\sqrt{-1}$ is substituted for x. But $P = 0$ and $Q = 0$;

$$\therefore \qquad P - Q\sqrt{-1} = 0 \quad . \quad . \quad . \quad (5).$$

Therefore $a - b\sqrt{-1}$ is a root of (1).

COR. 1.—An equation of an odd degree whose coefficients are real has at least one real root.

COR. 2.—The product of the two roots $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ is $a^2 + b^2$, which is a *real positive* quantity; hence, an equation of an even degree whose coefficients are real, and whose absolute term is negative, must have at least two real roots.

COR. 3.—The product of $x-(a+b\sqrt{-1})$ and $x-(a-b\sqrt{-1})$ is $(x-a)^2 + b^2$, which is a *rational* quadratic expression, and positive for all real values of x.

COR. 4.—If $a + \sqrt{b}$, in which $\sqrt{b}$ is a simple quadratic surd, is a root of an equation whose coefficients are *rational*, then will $a - \sqrt{b}$ be a root of that equation.

EXAMPLES.

1. $1-2\sqrt{-1}$ is a root of the equation $x^3-x^2+3x+5=0$; what are the other roots? *Ans.* -1 and $1+2\sqrt{-1}$.

2. $\sqrt{-1}$ is a root of the equation $x^4+4x^3+6x^2+4x+5=0$; what are the other roots?

3. $3+\sqrt{-2}$ is a root of the equation $x^4+x^3-25x^2+41x+66=0$; what are the other roots?

4. $\sqrt{2}$ is a root of the equation $x^4+2x^3-4x^2-4x+4=0$; what are the other roots?

5. $2+\sqrt{3}$ is a root of the equation $x^4-2x^3-5x^2-6x+2=0$; what are the other roots?

6. $\sqrt{3}$ and $1-2\sqrt{-1}$ are roots of the equation $x^5-x^4+8x^2-9x-15=0$; what are the other roots?

7. Has the equation $x^3-2x+4=0$ a real root? Why?

8. Has the equation $x^4-4x^3+4x-1=0$ any real roots? Why?

TRANSFORMATION OF EQUATIONS.

596. To transform an equation into another, the roots of which are those of the proposed equation with contrary signs.

Let r represent a root of the equation

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots\ldots=0 \quad \ldots \quad (1);$$

then

$$r^n+Ar^{n-1}+Br^{n-2}+Cr^{n-3}+\ldots\ldots=0 \quad \ldots \quad (2).$$

Changing the signs of (2),

$$-r^n-Ar^{n-1}-Br^{n-2}-Cr^{n-3}\ldots\ldots=0 \quad \ldots \quad (3).$$

Changing the signs of the alternate terms of (1),

$$x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+\ldots\ldots=0 \quad \ldots \quad (4).$$

Substituting $-r$ for x, the first member of (4) becomes

$$r^n + Ar^{n-1} + Br^{n-2} + Cr^{n-3} + \ldots,$$

or

$$-r^n - Ar^{n-1} - Br^{n-2} - Cr^{n-3} - \ldots.$$

according as n is *even* or *odd*. But, by (2) and (3), each of these expressions is equal to zero; hence $-r$ is a root of (4).

Since $-r$ is a root of (4), it is a root of the equation obtained by changing all the signs of (4); that is, $-r$ is a root of the equation

$$-x^n + Ax^{n-1} - Bx^{n-2} + Cx^{n-3} - \ldots = 0 \quad \ldots \quad (5).$$

Hence,

If the signs of the alternate terms of a complete equation be changed, the signs of all the roots will be changed.

An incomplete equation may be rendered complete by inserting the missing terms, with zero for the coefficient of each of them. Thus, by inserting $0x^4$ and $0x^2$, the equation $x^6 + 3x^5 - 4x^3 - 4x + 7 = 0$ becomes $x^6+3x^5+0x^4-4x^3+0x^2-4x+7=0$.

EXAMPLES.

1. The roots of the equation $x^3 - 7x^2 + 13x - 3 = 0$ are 3, $2 + \sqrt{3}$, and $2 - \sqrt{3}$; find the equation whose roots are -3, $-2 - \sqrt{3}$, and $-2 + \sqrt{3}$.

Ans. $x^3 + 7x^2 + 13x + 3 = 0$.

2. The roots of the equation $x^4 - 3x^3 + 3x^2 + 17x - 18 = 0$ are 1, -2, $2 + \sqrt{-5}$, and $2 - \sqrt{-5}$; what are the roots of the equation $x^4 + 3x^3 + 3x^2 - 17x - 18 = 0$?

Ans. -1, 2, $-2 - \sqrt{-5}$, $-2 + \sqrt{-5}$.

3. The roots of the equation $x^4 + 4x^3 - x^2 - 16x - 12 = 0$ are 2, -1, -2, and -3; what are the roots of the equation $-x^4 + 4x^3 + x^2 - 16x + 12 = 0$? *Ans.* -2, 1, 2, 3.

4. The roots of the equation $x^3 - 1 = 0$ are 1, $\frac{1}{2}(-1 + \sqrt{-3})$, and $\frac{1}{2}(-1 - \sqrt{-3})$; what are the roots of the equation $x^3 + 1 = 0$?

Ans. -1, $-\frac{1}{2}(-1 + \sqrt{-3})$, $-\frac{1}{2}(-1 - \sqrt{-3})$.

597. To transform an equation containing fractional coefficients into another in which the coefficients are integers, that of the first term being unity.

Let the proposed equation be

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Kx + L = 0 \quad . \quad . \quad . \quad (1),$$

in which some or all of the coefficients A, B, C, ... are supposed to be fractional.

Assume $y = kx$, or $x = \frac{y}{k}$. Substituting $\frac{y}{k}$ for x in (1) and multiplying the resulting equation by k^n,

$$y^n + Aky^{n-1} + Bk^2y^{n-2} + \ldots\ldots + Kk^{n-1}y + Lk^n = 0 \quad . \quad . \quad . \quad (2).$$

Now, since k is arbitrary, we may give it such a value as will make the coefficients Ak, Bk^2, ... Kk^{n-1}, Lk^n integers.

EXAMPLES.

Transform each of the following equations into another in which the coefficients are entire, that of the first term being unity:

1. $x^4 + \frac{a}{b}x^3 + \frac{c}{d}x^2 + \frac{e}{f}x + \frac{g}{h} = 0 \quad . \quad . \quad . \quad (1).$

Substituting $\frac{y}{k}$ for x in (1) and multiplying the resulting equation by k^4,

$$y^4 + \frac{ak}{b}y^3 + \frac{ck^2}{d}y^2 + \frac{ek^3}{f}y + \frac{gk^4}{h} = 0 \quad . \quad . \quad . \quad (2).$$

Assuming $k = bdfh$, (2) becomes

$$y^4 + adfhy^3 + cb^2df^2h^2y^2 + eb^3d^3f^2h^3y + gb^4d^4f^4h^3 = 0 \quad . \quad . \quad . \quad (3).$$

2. $x^3 + \frac{ax^2}{pm} + \frac{bx}{m} + \frac{c}{p} = 0 \quad . \quad . \quad . \quad (1).$

Substituting $\frac{y}{k}$ for x in (1) and multiplying the resulting equation by k^3,

$$y^3 + \frac{aky^2}{pm} + \frac{bk^2y}{m} + \frac{ck^3}{p} = 0 \quad . \quad . \quad . \quad (2).$$

Assuming $k = pm$, the L. C. M. of the denominators, (2) becomes

$$y^3 + ay^2 + bp^2my + cp^2m^3 = 0 \quad . \quad . \quad . \quad (3).$$

3. $x^4 - \frac{5}{6}x^3 + \frac{5}{12}x^2 - \frac{7}{150}x - \frac{13}{9000} = 0 \quad . \quad . \quad . \quad (1).$

Substituting $\frac{y}{k}$ for x in (1) and multiplying the resulting equation by k^4,

$$y^4 - \frac{5k}{6}y^3 + \frac{5k^2}{12}y^2 - \frac{7k^3}{150}y - \frac{13k^4}{9000} = 0 \quad . \quad . \quad . \quad (2).$$

Resolving the denominators in (2) into prime factors, we have

$6 = 2 \times 3, \quad 12 = 2^2 \times 3, \quad 150 = 2 \times 3 \times 5^2, \quad 9000 = 2^3 \times 3^2 \times 5^3.$

Assuming $k = 2 \times 3 \times 5$, (2) becomes

$$y^4 - \frac{5 \cdot 2 \cdot 3 \cdot 5}{2 \cdot 3}y^3 + \frac{5 \cdot 2^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 3}y^2 - \frac{7 \cdot 2^3 \cdot 3^3 \cdot 5^3}{2 \cdot 3 \cdot 5^2}y - \frac{13 \cdot 2^4 \cdot 3^4 \cdot 5^4}{2^3 \cdot 3^2 \cdot 5^3}$$
$$= 0 \quad . \quad . \quad . \quad (3).$$

Canceling common factors in (3),

$$y^4 - 5 \cdot 5y^3 - 5 \cdot 3 \cdot 5^2y^2 - 7 \cdot 2^2 \cdot 3^2 \cdot 5y - 13 \cdot 2 \cdot 3^2 \cdot 5 = 0;$$

that is, $y^4 - 25y^3 + 375y^2 - 1260y - 1170 = 0 \quad . \quad . \quad . \quad (4).$

If we had assumed $k = 9000$, which is the L. C. M. of the denominators of the given equation, the coefficients in the transformed equation would have been much larger than those in (4).

4. $x^3 - \frac{3}{35}x^2 + \frac{13}{2450}x - \frac{17}{68600} = 0.$

Ans. $y^3 - 6y^2 + 26y - 85 = 0.$

5. $x^5 - \frac{13}{12}x^4 + \frac{21}{40}x^3 - \frac{32}{225}x^2 - \frac{43}{600}x - \frac{1}{800} = 0.$

Ans. $y^5 - 65y^4 + 1890y^3 - 30720y^2 - 928800y - 972000 = 0.$

6. $x^3 - \frac{7}{3}x^2 + \frac{11}{36}x - \frac{25}{72} = 0.$

Ans. $y^3 - 14y^2 + 11y - 75 = 0.$

598. To transform an equation into another, the roots of which shall ~~differ from~~ those of the given equation by a given quantity h.

Let the proposed equation be

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Kx + L = 0 \quad . \quad . \quad . \quad (1).$$

Substituting $y + h$ for x in (1), we have

$$(y+h)^n + A(y+h)^{n-1} + B(y+h)^{n-2} + \ldots + K(y+h) + L = 0 \ldots (2).$$

Expanding and reducing, (2) becomes

$$\left.\begin{array}{l|l|ll}
y^n + nh & y^{n-1} + \dfrac{n(n-1)h^2}{2} & y^{n-2} + \ldots\ldots & + h^n \\
\quad + A & \quad + (n-1)Ah & & + Ah^{n-1} \\
 & \qquad\quad + B & & + Bh^{n-2} \\
 & & & + \ldots\ldots \\
 & & & + Kh \\
 & & & + L
\end{array}\right\} = 0 \ . \ . \ . \ (3).$$

The roots of (3) differ from those of (1) by h, for $x = y + h$.

Denoting the coefficient of y^{n-1} by A', that of y^{n-2} by $B', \ldots\ldots,$ and the independent term by L', (3) becomes

$$y^n + A'y^{n-1} + B'y^{n-2} + \ldots\ldots + J'y^2 + K'y + L' = 0 \quad . \quad . \quad . \quad (4).$$

We now propose to show that (4) may be deduced from (1) by *Synthetic Division.*

Restoring the value of y, (4) becomes

$$(x-h)^n + A'(x-h)^{n-1} + B'(x-h)^{n-2} + \ldots\ldots + J'(x-h)^2 + K'(x-h) + L' = 0 \quad . \quad . \quad . \quad (5).$$

Now the first member of (5) is identical with the first member of (1); for, in deducing (5) from (4), we merely retraced the steps by which (4) was deduced from (1). Hence the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Jx^2 + Kx + L = (x-h)^n + A'(x-h)^{n-1} + B'(x-h)^{n-2} + \ldots\ldots + J'(x-h)^2 + K'(x-h) + L' \quad . \quad . \quad . \quad (6)$$

is an identity.

Dividing the second member of (6) by $x - h$, we obtain the remainder L'; dividing the quotient by $x - h$, we obtain the remainder K'; dividing the second quotient by $x - h$, we obtain the remainder J'; and so on; hence if we treat the first member of (6) or the first member of (1) in the same way, we shall obtain the same remainders. But these successive remainders are the coefficients of (4). Hence the coefficients of (4) may be obtained from (1) by the following

RULE.

Divide the first member of (1) *by* $x - h$, *continuing the operation until a remainder is obtained which is independent of* x; *then divide the quotient by the same divisor, and so on, until* n *divisions have been performed: the successive remainders will be the coefficients of* (4).

EXAMPLES.

1. Find an equation whose roots are less by 2 than those of the equation $x^4 - 4x^3 - 8x + 32 = 0$.

Substituting $y + 2$ for x in this equation, we obtain $y^4 + 4y^3 - 24y = 0$, which is the equation required. The same result
making [illegible] or more concisely,
may be obtained by *Synthetic Division*, as follows:

$$\begin{array}{rrrrrl}
1 & -4 & +0 & -8 & +32 & \mid 2 \\
 & +2 & -4 & -8 & -32 & \mid \\
\hline
 & -2 & -4 & -16 & +0 & \text{1st rem.} \\
 & +2 & +0 & -8 & & \\
\hline
 & +0 & -4 & -24 & & \text{2d rem.} \\
 & +2 & +4 & & & \\
\hline
 & +2 & +0 & & & \text{3d rem.} \\
 & +2 & & & & \\
\hline
 & +4 & & & & \text{4th rem.}
\end{array}$$

Hence the required equation is $y^4 + 4y^3 + 0y^2 - 24y + 0 = 0$.

2. Find an equation whose roots are greater by 3 than those of the equation $x^4 + 16x^3 + 99x^2 + 228x + 144 = 0$.

Substituting $y - 3$ for x in this equation, we obtain $y^4 + 14y^3 + 9y^2 - 42y = 0$. The same result may be obtained by Synthetic Division, as follows:

$$\begin{array}{rrrrr|l}
1 + 16 & + 99 & + 228 & + 144 & & -3 \\
- \ \ 3 & - 39 & - 180 & - 144 & & \\
\hline
+ 13 & + 60 & + \ \ 48 & + \ \ \ 0 & \text{1st rem.} & \\
- \ \ 3 & - 30 & - \ \ 90 & & & \\
\hline
+ 10 & + 30 & - \ \ 42 & \text{2d rem.} & & \\
- \ \ 3 & - 21 & & & & \\
\hline
+ \ \ 7 & + \ \ 9 & \text{3d rem.} & & & \\
- \ \ 3 & & & & & \\
\hline
+ \ \ 4 & \text{4th rem.} & & & &
\end{array}$$

Hence the required equation is $y^4 + 4y^3 + 9y^2 - 42y = 0$.

3. Find an equation whose roots are less by 2 than those of the equation $x^4 - 4x^3 - 8x + 32 = 0$. *Ans.* $y^4 + 4y^3 - 24y = 0$.

4. Find an equation whose roots are less by 3 than those of the equation $x^4 - 12x^3 + 17x^2 - 9x + 7 = 0$.

Ans. $y^4 - 37y^2 - 123y - 110 = 0$.

599. To transform an equation into another in which the second or third term shall not appear.

Since h in equation (3) of Art. **598** is arbitrary, we may give to it such a value as will cause the second term of that equation to vanish.

Assume $nh + \mathrm{A} = 0$; then $h = -\dfrac{\mathrm{A}}{n}$. Substituting this value for h in (3), we obtain an equation of the form of

$$y^n + \mathrm{B}'y^{n-2} + \mathrm{C}'y^{n-3} + \ldots\ldots + \mathrm{K}'y + \mathrm{L}' = 0.$$

If we assume $\dfrac{n(n-1)h^2}{2} + (n-1)\mathrm{A}h + \mathrm{B} = 0$, the third term of (3) will disappear.

COR.—The value of h which makes the second term disappear may cause the disappearance of the third or some other term.

In order that the third term may disappear at the same time with the second, it is necessary that the value of h which satisfies the equation $nh + A = 0$ shall also satisfy the equation $\frac{n(n-1)h^2}{2} + (n-1)Ah + B = 0$. Substituting $-\frac{A}{n}$ for h in this equation, we have $\frac{n(n-1)}{2} \cdot \frac{A^2}{n^2} - (n-1)\frac{A^2}{n} + B = 0$; whence $A^2 = \frac{2nB}{n-1}$. This equation expresses the relation which must subsist between the coefficients A and B in order that the third term may disappear with the second.

EXAMPLES.

1. Transform the equation $x^3 - 6x^2 + 8x - 2 = 0$ into another wanting the second term. *Ans.* $y^3 - 4y - 2 = 0$.

2. Transform the equation $x^4 - 12x^3 + 17x^2 - 9x + 7 = 0$ into another wanting the second term.
Ans. $y^4 - 37y^2 - 123y - 110 = 0$.

3. Transform the equation $x^3 - 6x^2 + 13x - 12 = 0$ into another wanting the second term. *Ans.* $y^3 + y - 2 = 0$.

4. Transform the equation $x^3 + 5x^2 + 8x - 1 = 0$ into two others, each wanting the third term.
Ans. $y^3 - y^2 - 5 = 0$ and $y^3 + y^2 - \frac{112}{27} = 0$.

5. Can the equation $x^3 + 6x^2 + 12x - 56 = 0$ be transformed into another wanting the second and third terms?

THEOREM OF DESCARTES.

600. In any series of quantities a pair of consecutive like signs is called a ***Permanence*** of signs, and a pair of consecutive unlike signs is called a ***Variation*** of signs. Thus, in the expression $x^8 - 3x^7 - 4x^6 + 7x^5 + 3x^4 + 2x^3 - x^2 - x + 1$, there are four permanences and four variations.

601. If the equation $f(x) = 0$ is complete, the sum of the number of permanences and the number of variations in the signs of the terms of $f(x)$ is equal to the greatest exponent of x in the equation.

602. *Theorem of Descartes.*—*The number of real positive roots of the equation $f(x) = 0$ cannot exceed the number of variations in the signs of its terms; and, if the equation $f(x) = 0$ is complete, the number of real negative roots cannot exceed the number of permanences in the signs of its terms.*

Represent the real positive roots of the equation

$$f(x) = 0 \quad . \quad . \quad . \quad (1)$$

by $a, b, c \ldots .$, and suppose (1) to be divided by the product of all the factors $x - a, x - b, x - c, \ldots .$ corresponding to the real positive roots (**586**). Represent the resulting equation by

$$f_1(x) = 0 \quad . \quad . \quad . \quad (2).$$

This equation has no real positive roots.

We shall now show that if (2) be multiplied by the factor $x - a$ corresponding to a real positive root, the number of variations in the resulting equation will be at least one greater thau in (2).

I. Suppose (2) to be complete, and let the signs of its terms be

$$\begin{array}{lccccccc}
 & + & + & - & - & - & + . & \\
\text{The signs of the multiplier are} & + & - . & & & & & \\
\hline
 & + & + & - & - & - & + . & \\
 & & - & - & + & + & + & - . \\
\hline
\text{The signs of the product are} & + & \pm & - & \mp & \mp & + & - .
\end{array}$$

A double sign is placed where the sign of any term in the product is ambiguous.

Now, taking the ambiguous signs as we please, the number of variations in the product is greater than in the multiplicand; and this is still true if we suppose some or all of the terms having ambiguous signs to vauish.

II. If (2) is incomplete, reduce it to a complete form by inserting the missing terms with zero for the coefficient of each; the resulting equation will contain at least as many variations as (2). Multiplying the completed equation by $x - a$, the number of variations in the product will be greater than in the multiplicand (I). But the product thus obtained is the same as the pro-

duct of $f_1(x)$ and $x - a$; hence, the number of variations in the product of $f_1(x)$ and $x - a$ is greater than in $f_1(x)$.

We have thus shown that when the factor $x - a$ is introduced into (2), the resulting equation contains at least one more variation than (2). In like manner it may be shown that when the factor $x - b$ is introduced into the resulting equation, at least one more variation is introduced; and so on.

Hence the number of real positive roots of the equation $f(x) = 0$ cannot exceed the number of variations in the signs of its terms.

We prove the second part of the theorem as follows:

Suppose (1) to be complete, and let the signs of its alternate terms be changed; then the signs of the roots will be changed (**596**), the permanences will become variations, and the variations will become permanences. But the number of real positive roots of the resulting equation cannot exceed the number of variations in the signs of its terms; hence the number of real negative roots of the given equation cannot exceed the number of permanences in the signs of its terms.

Cor. 1.—Whether the equation $f(x) = 0$ is complete or not, its roots are numerically equal to those of the equation $f(-x) = 0$; but the signs of the two sets of roots are opposite. Hence the number of real negative roots of the equation $f(x) = 0$ is equal to the number of real positive roots of the equation $f(-x) = 0$. But the number of real positive roots of the equation $f(-x) = 0$ cannot exceed the number of variations in the signs of its terms. We may therefore state the theorem of Descartes as follows:

The number of real positive roots of the equation $f(x) = 0$ cannot exceed the number of variations in the signs of $f(x)$, and the number of its real negative roots cannot exceed the number of variations in the signs of $f(-x)$.

Illustration.—The equation $x^4 + 3x^2 + 5x - 7 = 0$ has only one variation of signs; therefore it cannot have more than one real positive root. By putting $-x$ in the place of x, we obtain the equation $x^4 + 3x^2 - 5x - 7 = 0$. This equation has only one variation of signs; therefore it cannot have more than one

real positive root; hence the original equation cannot have more than one real negative root.

COR. 2.—If the equation $f(x) = 0$ is complete, and all its roots are real, the number of positive roots is equal to the number of variations in the signs of its terms, and the number of negative roots is equal to the number of permanences in the signs of its terms.

Denoting the number of permanences by p, the number of variations by v, the number of positive roots by P, the number of negative roots by N, and the highest exponent of x in $f(x)$ by n, we have $v + p = n$ and $P + N = n$; hence $v + p = P + N$. Now P cannot exceed v, and N cannot exceed p; hence $P = v$, and $N = p$.

COR. 3.—By means of the theorem of Descartes we can sometimes detect the presence of imaginary roots in an equation.

Illustration.—The equation $x^2 + 16 = 0$ has no variation of signs; therefore it has no real positive root. By putting $-x$ in the place of x, we obtain the equation $x^2 + 16 = 0$. This equation has no variation of signs; therefore it has no real positive root; hence the original equation has no real negative root. Therefore the roots of the equation $x^2 + 16 = 0$ are imaginary.

EXAMPLES.

1. Show that the equation $x^3 + 5x + 18 = 0$ has only one real root.

2. All the roots of the equation $x^3 + 5x^2 + 2x - 8 = 0$ are real; how many of them are negative? *Ans.* Two.

3. All the roots of the equation $x^4 - 5x^3 - 7x^2 + 29x + 30 = 0$ are real; how many of them are positive? *Ans.* Two.

4. All the roots of the equation $x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0$ are real; how many of them are positive? *Ans.* Three.

DERIVED FUNCTIONS.

603. Substituting $x+h$ for x in the identity

$$f(x) = x^n + Ax^{n-1} + Bx^{n-2} + \dots + Kx + L \quad . \quad . \quad . \quad (1),$$

and arranging the result according to the ascending powers of h, we obtain

$$f(x+h)$$

$$\begin{array}{llll} = \quad x^n + & nx^{n-1} \Big| h & + n(n-1)x^{n-2} \Big| \dfrac{h^2}{1\cdot 2} + \dots & \dots (2). \\ + Ax^{n-1} + & (n-1)Ax^{n-2} & + (n-1)(n-2)Ax^{n-3} & \\ + Bx^{n-2} + & (n-2)Bx^{n-3} & + (n-2)(n-3)Bx^{n-4} & \\ \dots\dots & \dots\dots\dots & \dots\dots\dots\dots\dots & \\ + Kx & + K & & \\ + L & & & \end{array}$$

Denoting the coefficient of h by $f'(x)$, that of $\dfrac{h^2}{1.2}$ by $f''(x)$, and so on, (2) may be written

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{\underline{|2}} + f'''(x)\frac{h^3}{\underline{|3}} + \dots \quad . \quad . \quad . \quad (3).$$

The expression $f(x)$ is called the *primitive function*, the expression $f'(x)$ is called the *first derived function*, or simply the *first derivative*, the expression $f''(x)$ is called the *second derivative*, and so on.

The first derivative may be obtained from the primitive function by multiplying each of its terms by the exponent of x in that term and dividing the result by x; the second derivative may be obtained from the first in the same way that the first is obtained from the primitive function; and so on.

EXAMPLES.

1. Find the derivatives of $x^3 - 6x^2 + 8x - 2$.

$$Ans. \begin{cases} \text{1st.} & 3x^2 - 12x + 8, \\ \text{2d.} & 6x - 12, \\ \text{3d.} & 6. \end{cases}$$

2. Transform the equation $x^3-6x^2+8x-2=0$ into another wanting the second term.

Substituting $x+2$ for x in the identity $f(x)=x^3-6x^2+8x-2$, we have

$$f(x+2)=x^3-6x^2+8x-2+(3x^2-12x+8)2+(6x-12)\frac{2^2}{\lfloor 2}+6\frac{2^3}{\lfloor 3}$$

$=x^3-4x-2$; hence the required equation is $x^3-4x-2=0$.

3. Find an equation whose roots are less by 1 than those of the equation $x^3-2x^2+3x-4=0$.

Substituting $x+1$ for x in the identity $f(x)=x^3-2x^2+3x-4$, we have

$$f(x+1)=x^3-2x^2+3x-4+(3x^2-4x+3)1+(6x-4)\frac{1^2}{\lfloor 2}+6\frac{1^3}{\lfloor 3}$$

$=x^3+x^2+2x-2$; hence the required equation is

$$x^3+x^2+2x-2=0.$$

Let the student solve all the examples of Art. **599** by the method of *derived functions.*

604. *The first derivative of the product of two functions of the same quantity is equal to the sum of the products obtained by multiplying each by the first derivative of the other.*

Substituting $x+h$ for x in the two expressions $f(x)$ and $f_1(x)$, we obtain

$$f(x+h)=f(x)+f'(x)h+\dots \quad . \quad . \quad . \quad (1) \quad \textbf{(603)},$$

and $$f_1(x+h)=f_1(x)+f_1'(x)h+\dots \quad . \quad . \quad . \quad (2).$$

Multiplying (1) by (2),

$$f(x+h)f_1(x+h)=f(x)f_1(x)+f_1(x)f'(x)h+f(x)f_1'(x)h+\dots$$

$$=f(x)f_1(x)+[f_1(x)f'(x)+f(x)f_1'(x)]h+\dots \quad . \quad . \quad . \quad (3).$$

The coefficient of h in (3) is the sum of the products obtained by multiplying $f_1(x)$ by the first derivative of $f(x)$ and $f(x)$ by the first derivative of $f_1(x)$ and this coefficient is the first derivative of $f(x)f_1(x)$ **(603)**.

COR.—In like manner it may be shown that the first derivative of the product of three or more functions of the same quantity is equal to the sum of the products obtained by multiplying the first derivative of each by the product of the other functions.

EXAMPLES.

Find the first derivative of each of the following expressions:

1. $x^2(x-a)^3$. *Ans.* $2x(x-a)^3 + 3(x-a)^2x^2$.

2. $(a+x)(b+x)$. *Ans.* $b+x+a+x = a+b+2x$.

3. $(x-a)^2(x-b)^3$. *Ans.* $2(x-a)(x-b)^3+3(x-b)^2(x-a)^2$.

4. $(x-a)^3(x-b)^4(x-c)^5$.
Ans. $3(x-a)^2(x-b)^4(x-c)^5+4(x-b)^3(x-a)^3(x-c)^5$ $+5(x-c)^4(x-a)^3(x-b)^4$.

5. $(x-a)^n(x-b)^m$.
Ans. $n(x-a)^{n-1}(x-b)^m+m(x-b)^{m-1}(x-a)^n$.

ROOTS COMMON TO TWO EQUATIONS.

605. If a is a root of the equation $f(x)=0$, $f(x)$ is divisible by $x-a$, and if a is a root of the equation $f_1(x)=0$, $f_1(x)$ is divisible by $x-a$; hence the roots of the equation obtained by putting the G. C. D. of $f(x)$ and $f_1(x)$ equal to zero will be the roots common to the two equations $f(x)=0$ and $f_1(x)=0$.

EXAMPLES.

1. Find the root which is common to the two equations

$$x^4-2x^3-7x^2+20x-12=0 \quad \text{and} \quad 4x^3-6x^2-14x+20=0.$$

The G. C. D. of the first members of these equations is $x-2$; hence 2 is a root common to the given equations.

2. Find the roots common to the two equations

$$x^4-2x^3-11x^2+12x+36=0 \quad \text{and} \quad 4x^3-6x^2-22x+12=0.$$

Ans. 3 and -2.

3. Find the roots common to the two equations $x^7-3x^6+x^5-4x^2+12x-4=0$ and $2x^4-6x^3+3x^2-3x+1=0$.

Ans. $\frac{3 \pm \sqrt{5}}{2}$.

4. How many roots are common to the two equations $x^6-49x^4+67x^3+10x^2-25x-4=0$ and $2x^5-18x^4+39x^3-25x^2+x+1=0$?

Ans. Three.

EQUAL ROOTS.

606. The equation $f(x)=0$ is called the *Primitive Equation*, and the equation $f'(x)=0$, which is obtained by putting the first derivative of $f(x)$ equal to zero, is called the *First Derived Equation.*

607. *If a root occurs n times in the equation $f(x)=0$, it will occur $n-1$ times in the equation $f'(x)=0$.*

Let the proposed equation be

$$f(x)=(x-a)^n(x-b)(x-c)\ldots\ldots=0 \quad . \quad . \quad . \quad (1),$$

in which a occurs as a root n times.

The first derivative of each of the factors $x-b$, $x-c$, ... is 1; hence the first derived equation is

$$f'(x)=n(x-a)^{n-1}(x-b)(x-c)\ldots\ldots+(x-a)^n(x-c)\ldots\ldots$$
$$+(x-a)^n(x-b)\ldots\ldots+=0 \quad . \quad . \quad . \quad (2),$$

in which a occurs as a root $n-1$ times (**587**).

Cor.—A root which occurs only once in (1) does not occur in (2); hence any root which is common to (1) and (2) is one of the equal roots of (1).

EXAMPLES.

Find all the roots of each of the following equations:

1. $f(x)=x^3-7x^2+16x-12=0$.

$f'(x)=3x^2-14x+16=0$. The G. C. D. of $f(x)$ and $f'(x)$ is $x-2$. Putting this equal to zero, we have $x-2=0$; whence $x=2$. The given equation, therefore, has two roots equal to 2. The remaining root of the given equation may be found by the principle of Art. **588**, Cor.

2. $x^4 - 11x^3 + 44x^2 - 76x + 48 = 0$. *Ans.* 2, 2, 3, 4.

3. $2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0$. *Ans.* $3, 3, \pm\sqrt{-\frac{1}{2}}$.

4. $x^6 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0$.

Ans. $1, 1, 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{-11}$.

5. $f(x) = x^7 - 9x^6 + 6x^4 + 15x^3 - 12x^2 - 7x + 6 = 0 \quad . \quad . \quad . \quad (1).$

The first derived equation is

$$f'(x) = 7x^6 - 45x^4 + 24x^3 + 45x^2 - 24x - 7 = 0 \quad . \quad . \quad . \quad (2).$$

The G. C. D. of $f(x)$ and $f'(x)$ is $x^3 - x^2 - x + 1$. Equating this with zero, we have

$$x^3 - x^2 - x + 1 = 0 \quad . \quad . \quad . \quad (3).$$

The G. C. D. of the first member of (3) and its first derivative is $x - 1$. Equating this with zero, we find $x = 1$; hence (3) has two roots equal to 1. The remaining root of (3) is -1 (**593**).

Now, since (3) has two roots equal to 1, and one root equal to -1, (1) must have three roots equal to 1 and two roots equal to -1. Dividing $f(x)$ by $(x-1)^3(x+1)^2$, we obtain $x^2 + x - 6$. Equating this with zero, we have $x^2 + x - 6 = 0$; whence, $x = 2$ or -3. The roots of (1) are therefore $1, 1, 1, -1, -1, 2, -3$.

6. $x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2 = 0$.

7. $x^6 - 6x^4 + 4x^3 + 9x^2 - 12x + 4 = 0$.

LIMITS OF THE ROOTS OF AN EQUATION.

608. *If the coefficients of $f(x)$ are real, and the results obtained by substituting p and q for x in $f(x)$ have like signs, the equation $f(x) = 0$ has either no root or an even number of roots lying between p and q; but if the results have contrary signs, the equation has an odd number of roots lying between p and q.*

Let the real roots of the equation $f(x) = 0 \ . \ . \ . \ (1)$ be denoted by $a, b, c, \ldots . k$, and let the quotient obtained by dividing $f(x)$ by $(x-a)(x-b)(x-c) \ldots . (x-k)$ be denoted by $f_1(x)$; then

$$f(x) = (x-a)(x-b)(x-c) \ldots . (x-k) f_1(x) \quad . \ . \ . \quad (2).$$

Now, since $f_1(x)$ is the product of all the factors corresponding to the imaginary roots of (1), and since the number of these imaginary roots is even, it follows that $f_1(x)$ is positive for all real values of x (**595,** Cor. 3).

Substituting p and q, in succession, for x in (2), we have

$$f(p) = (p-a)(p-b)(p-c) \ldots . (p-k) f_1(p) \quad . \ . \ . \quad (3),$$

$$f(q) = (q-a)(q-b)(q-c) \ldots . (q-k) f_1(q) \quad . \ . \ . \quad (4).$$

Dividing (3) by (4),

$$\frac{f(p)}{f(q)} = \frac{p-a}{q-a} \cdot \frac{p-b}{q-b} \cdot \frac{p-c}{q-c} \ldots . \frac{p-k}{q-k} \cdot \frac{f_1(p)}{f_1(q)} \quad . \ . \ . \quad (5).$$

Suppose $f(p)$ and $f(q)$ have like signs; then $\frac{f(p)}{f(q)}$ will be positive; and since $\frac{f_1(p)}{f_1(q)}$ is positive, either all the factors $\frac{p-a}{q-a}, \frac{p-b}{q-b}, \frac{p-c}{q-c}, \ldots ., \frac{p-k}{q-k}$ must be positive, or the number of negative factors must be even. If all the factors are positive, no root of (1) can lie between p and q; for, if possible, suppose the root b lies between p and q; then $p-b$ and $q-b$ would have contrary signs; therefore $\frac{p-b}{q-b}$ would be negative.

If the number of negative factors is even, then (1) has an even number of its roots lying between p and q; for, if any factor, as $\frac{p-c}{q-c}$, is negative, then $p-c$ and $q-c$ must have contrary signs; therefore c lies between p and q.

Suppose $f(p)$ and $f(q)$ have contrary signs; then $\frac{f(p)}{f(q)}$ will be negative, and the number of negative factors must be odd.

But when any factor as $\dfrac{p-c}{q-c}$ is negative, the root c lies between p and q; hence (1) has an odd number of its roots between p and q when $f(p)$ and $f(q)$ have contrary signs.

COR. 1.—If p is less than the least root of the equation $f(x)=0$, $f(p)$ will be positive or negative, according as the degree of the equation is even or odd.

COR. 2.—If q is greater than the greatest root of the equation $f(x)=0$, $f(q)$ will be positive.

COR. 3.—Suppose that (1) has no equal roots, and that a is the smallest of its real roots, b the next smallest, and so on. Now suppose x to assume, in succession, every possible value from $-\infty$ to $+\infty$; then the sign of $f(x)$ will change from $+$ to $-$, or from $-$ to $+$, as often as x passes a real root of the equation; for as long as x is less than a, all the factors $x-a$, $x-b$, $x-c$, ... $x-k$ are negative; but when x becomes greater than a and less than b, the factor $x-a$ will be positive, while the other factors $x-b$, $x-c$, ... $x-k$ will be negative. In like manner it may be shown that the sign of $f(x)$ changes when x passes either of the other real roots.

EXAMPLES.

1. Find the first figure of one of the roots of the equation $f(x)=x^3+x^2+x-100=0$.

When $x=4$, $f(x)$ is negative; and when $x=5$, $f(x)$ is positive; hence there must be a root between 4 and 5; that is, 4 is the first figure of one of the roots.

2. Find the first figure of each of the roots of the equation $x^3-3x^2-12x+24=0$. *Ans.* 1, 4, -3.

3. Find the first figure of each of the roots of the equation $x^4-12x^2+12x-3=0$. *Ans.* 2, .6, .4, -3.

4. Find the first figure of each of the roots of the equation $x^5-10x^3+6x+1=0$. *Ans.* -3, $-.6$, $-.1$, .8, 3.

609. **To find a superior limit of the positive roots of an equation.**

Suppose the equation to be of the n^{th} degree. Denote the negative coefficient whose absolute value is the greatest by $-P$, the exponent of x in the negative term of highest degree by m, and the absolute value of the sum of all the negative terms by N; then

$$N < P + Px + Px^2 + \dots + Px^m \quad . \quad . \quad . \quad (1).$$

But $P + Px + Px^2 + \dots + Px^m = \dfrac{Px^{m+1} - P}{x-1} \quad . \quad . \quad . \quad (2)$ (**520**);

$$\therefore \qquad N < \frac{Px^{m+1} - P}{x-1} < \frac{Px^{m+1}}{x-1} \quad . \quad . \quad . \quad (3).$$

Now, the absolute value of the sum of all the negative terms of the given equation is equal to the sum of all the positive terms; hence $\dfrac{Px^{m+1}}{x-1}$ must be greater than any positive term as x^n; that is,

$$x^n < \frac{Px^{m+1}}{x-1} \quad . \quad . \quad . \quad (4).$$

Multiplying both members of (4) by $\dfrac{x-1}{x^{m+1}}$,

$$(x-1)\,x^{n-m-1} < P \quad . \quad . \quad . \quad (5).$$

But $x - 1 < x$; hence, if $x > \frac{1}{2}$,

$$(x-1)^{n-m-1} < x^{n-m-1} \quad . \quad . \quad . \quad (6).$$

Multiplying both members of (6) by $x - 1$,

$$(x-1)^{n-m} < (x-1)x^{n-m-1} \quad . \quad . \quad . \quad (7);$$

$$\therefore \qquad (x-1)^{n-m} < P \quad . \quad . \quad . \quad (8);$$

whence, $$x - 1 < \sqrt[n-m]{P},$$

or $$x < 1 + \sqrt[n-m]{P}. \text{ (9) if } x > \tfrac{1}{2};$$

but in [illegible] (1),

Denoting this superior limit of the positive roots by L, we have

$$L = 1 + \sqrt[n-m]{P}.$$

EXAMPLES.

Find a superior limit of the positive roots in each of the following equations:

1. $x^5 + 5x^4 + 2x^3 - 14x^2 - 26x + 10 = 0.$

In this equation $n = 5$, $m = 2$, and $P = 26$; hence,

$$L = 1 + \sqrt[3]{26}.$$

2. $x^4 + 5x^3 - 25x^2 - 12x + 68 = 0.$ *Ans.* 6.
3. $x^4 - 5x^2 - 9x + 12 = 0.$ *Ans.* 4.
4. $x^3 + x^2 + 3x - 8 = 0.$ *Ans.* 3.

610. To find an inferior limit of the positive roots of an equation.

Substitute $\frac{1}{y}$ for x iu the given equation, and find a superior limit of the positive values of y in the resulting equation. Denote this limit by L'; then

$$y < L';$$

whence,
$$\frac{1}{y} > \frac{1}{L'};$$

that is,
$$x > \frac{1}{L'}.$$

Hence $\frac{1}{L'}$ is an inferior limit of the positive roots of the given equation.

EXAMPLES.

Find an inferior limit of the positive roots in each of the following equations:

1. $x^5 + 5x^4 + 2x^3 - 14x^2 - 26x + 10 = 0.$

Substituting $\frac{1}{y}$ for x and reducing, we have

$$y^5 - \frac{26}{10}y^4 - \frac{14}{10}y^3 + \frac{2}{10}y^2 + \frac{5}{10}y + \frac{1}{10} = 0.$$

A superior limit of the positive roots of this equation is 3.6; hence $\frac{1}{3.6}$ is an inferior limit of the positive roots of the given equation.

2. $x^5 + 5x^4 - 20x^2 - 19x - 2 = 0.$ *Ans.* $\dfrac{2}{2+\sqrt[4]{40}}$.

3. $x^6 - 5x^5 + x^4 + 12x^3 - 12x^2 + 1 = 0.$

4. $x^4 - 8x^3 + 12x^2 + 16x - 39 = 0.$

611. To find the limits of the negative roots of an equation.

Substitute $-x$ for x in the given equation, and find the limits of the positive roots of the resulting equation. By changing the signs of these limits we obtain the limits of the negative roots of the given equation (**602**, Cor. 1).

EXAMPLES.

Find the limits of the negative roots in each of the following equations:

1. $x^3 - 3x^2 + 5x + 7 = 0.$ *Ans.* $-(1 + \sqrt[3]{7}),\ -\dfrac{7}{12}$.

2. $x^4 - 15x^2 - 10x + 24 = 0.$

3. $x^6 - 3x^5 + 2x^4 + 27x^3 - 4x^2 - 1 = 0.$

STURM'S THEOREM.

612. If the coefficients of $f(x)$ are real and the equation $f(x) = 0$ has no equal roots, then, if x is made to assume, in succession, all real values from $-\infty$ to $+\infty$, the sign of $f(x)$ will change as often as x passes a real root of the equation (**608**, Cor. 3). Sturm's Theorem enables us to determine the number of such changes of sign.

613. ***Sturm's Functions.***—Let $f(x) = 0$ be an equation whose coefficients are real, and which is freed from equal roots (**607**); and let $f'(x)$ be the first derivative of $f(x)$.

We now apply to $f(x)$ and $f'(x)$ the process of finding their G. C. D. (**125**), with this modification, namely: 1. *When a remainder is found which is of a lower degree than the corresponding dividend and divisor, we change its sign and use the result for the next divisor.* 2. *We neither introduce nor reject a negative factor in preparing for division.*

We continue the operation until a remainder is obtained which is independent of x, and change the sign of that remainder also.

Let $f_1(x)$, $f_2(x)$, $f_3(x)$, ... $f_n(x)$ denote the series of modified remainders thus obtained.

The functions $f(x)$, $f'(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, ... $f_n(x)$ are called *Sturm's Functions.*

The functions $f'(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, ... $f_n(x)$ are called *Auxiliary Functions.*

614. *Sturm's Theorem.*—*If x be conceived to assume, in succession, all real values from $-\infty$ to $+\infty$, there will be no change in the number of variations in the signs of the series of functions $f(x)$, $f'(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, ... $f_n(x)$, except when x passes through a real root of the equation $f(x)=0$; and when x passes through such a root, there will be a loss of only one variation.*

I. $f_n(x)$ is not zero; for, by hypothesis, it is independent of x; hence, if it were zero, $f(x)$ and $f'(x)$ would have a common divisor, and the equation $f(x)=0$ would have equal roots (**607**); but this is contrary to the hypothesis.

II. Two consecutive functions cannot vanish for the same value of x.

Let $q_1, q_2, q_3, \ldots. q_n$ denote the successive quotients obtained by performing the operations described in Art. **613**; then, by the principles of division,

$$f(x)=q_1 f'(x)-f_1(x) \quad . \quad . \quad . \quad (1),$$
$$f'(x)=q_2 f_1(x)-f_2(x) \quad . \quad . \quad . \quad (2),$$
$$f_1(x)=q_3 f_2(x)-f_3(x) \quad . \quad . \quad . \quad (3),$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$f_{n-2}(x)=q_n f_{n-1}(x)-f_n(x) \quad . \quad . \quad . \quad (n).$$

Now suppose $f'(x)$ and $f_1(x)$ to vanish at the same time; then by (2) we shall have $f_2(x)=0$; hence by (3), $f_3(x)=0$; and so on; that is, if two consecutive functions vanish at the same time, all the succeeding functions, including $f_n(x)$ would vanish; but this is impossible (I).

III. When any auxiliary function vanishes, the two adjacent functions have contrary signs. Thus, if $f_2(x) = 0$, we have by (3), $f_1(x) = -f_3(x)$.

IV. No change can be made in the sign of any one of Sturm's functions, except when x passes through a value which causes that function to vanish (**608**, Cor. 3).

V. Sturm's functions neither gain nor lose a variation of signs when x passes through a value which causes one or more of the auxiliary functions to vanish, but which does not cause $f(x)$ to vanish.

1. Suppose $f_1(x)$ vanishes when $x = c$, and that no other function vanishes for this value of x. Let h be a positive quantity so small that no one of Sturm's functions except $f_1(x)$ vanishes while x is passing from $c - h$ to $c + h$.

When $x = c$, $f'(x)$ and $f_2(x)$ have contrary signs (III); hence they have contrary signs all the time that x is passing from $c - h$ to $c + h$ (IV). Now at the instant x becomes equal to c, $f_1(x)$ changes its sign (**608**, Cor. 3); hence, before the change, its sign is like that of one of the adjacent functions, and after the change it is like that of the other. But no change in the number of variations of signs in a row of signs can be made by simply changing a sign situated between two adjacent contrary signs. Thus, in the row of signs $+ - + - - + - - + -$ there are seven variations; and if we change the fourth sign there are still seven variations.

Hence Sturm's functions neither gain nor lose a variation of signs while x is passing from $c - h$ to $c + h$.

2. Suppose that when $f_1(x)$ vanishes, other auxiliary functions vanish. The vanishing functions cannot be consecutive (II); the functions adjacent to each vanishing function have contrary signs while x is passing from $c - h$ to $c + h$; and each vanishing function changes its sign at the instant x becomes equal to c. But, as we have just shown, this change of sign does not change the number of variations in the row of signs.

VI. Sturm's functions lose one variation of signs, and only one, each time x passes through a real root of the equation $f(x) = 0$. Let a be a real root of the equation $f(x) = 0$; then $f(a) = 0$.

Substituting $a + h$ for x in $f(x)$ and $f'(x)$, and developing by Art. **603**, we have

$$f(a+h) = h\left(f'(a) + f''(a)\frac{h}{\underline{|2}} + f'''(a)\frac{h^2}{\underline{|3}} + \cdots\right) \quad . \quad . \quad . \quad (1),$$

$$f'(a+h) = \quad f'(a) + f''(a)\,h + f'''(a)\frac{h^2}{\underline{|2}} + \cdots \quad . \quad . \quad . \quad (2).$$

Now assume the absolute value of h to be so small that the first term in each of these developments shall be numerically greater than the sum of the other terms; then the sign of $f(a+h)$ will be the same as that of $hf'(a)$, and the sign of $f'(a+h)$ will be the same as that of $f'(a)$. Hence $f(x)$ and $f'(x)$ will have contrary signs when h is negative, and like signs when h is positive. But when h is negative, x is less than a, and when h is positive, x is greater than a; hence when x passes a real root of the equation $f(x) = 0$, a variation is changed into a permanence. Now it is evident, from (2), that $f'(x)$ cannot vanish as long as h has such a value that $f'(a)$ is numerically greater than $f''(a)h + f'''(a)\dfrac{h^2}{\underline{|2}} + \cdots$. Some of the auxiliary functions lying between $f'(x)$ and $f_n(x)$ may, however, vanish and change signs while x is passing through the root a; but the change of a sign lying between two adjacent contrary signs (III) does not change the number of variations in the row of signs (V, 1). Therefore, when x passes through the root a, Sturm's functions lose one variation of signs, and only one.

In the same way it may be shown that when x passes through any other real root of the equation $f(x) = 0$, Sturm's functions lose another variation of signs.

Cor. 1.—The number of real roots of the equation $f(x) = 0$ is equal to the number of variations of signs lost by Sturm's functions while x is passing from $-\infty$ to $+\infty$; the number of real negative roots is equal to the number of variations of signs lost while x is passing from $-\infty$ to 0; and the number of real positive roots is equal to the number of variations of signs lost while x is passing from 0 to $+\infty$.

COR. 2.—Let a be the smallest real root of the equation $f(x)=0$, b the next greater, c the next, and so on.

Just after x passes through the root a, $f(x)$ and $f'(x)$ have like signs; and just before x passes through the root b, $f(x)$ and $f'(x)$ have contrary signs (VI). But $f(x)$ does not change its sign while x is passing from a to b; hence $f'(x)$ must change its sign. Therefore the equation $f'(x)=0$ has one real root between a and b. In the same way it may be shown that the equation $f'(x)=0$ has one real root between b and c. Therefore, between any two consecutive real roots of the equation $f(x)=0$ there is one real root of the equation $f'(x)=0$.

SCH.—The sign of each remainder is changed in order that there may be neither a gain nor a loss in the number of variations in the row of signs, except when x passes through a real root of the equation (III–V).

EXAMPLES.

Find the number and situation of the real roots of the following equations:

1. $x^3-3x^2-4x+13=0.$

$$\begin{aligned} f(x) &= x^3-3x^2-4x+13, \\ f'(x) &= 3x^2-6x-4 \quad (\mathbf{603}), \\ f_1(x) &= 2x-5 \quad (\mathbf{613}), \\ f_2(x) &= +1. \end{aligned}$$

ASSUMED VALUES OF x.	FUNCTIONS AND THEIR SIGNS. $f(x)$,	$f'(x)$,	$f_1(x)$,	$f_2(x)$.	NUMBER OF VARIATIONS.
$-\infty$	$-$	$+$	$-$	$+$	3
0	$+$	$-$	$-$	$+$	2
1	$+$	$-$	$-$	$+$	2
2	$+$	$-$	$-$	$+$	2
3	$+$	$+$	$+$	$+$	0
$+\infty$	$+$	$+$	$+$	$+$	0

Hence all the roots of the equation are real, two of them are positive and the other negative; and the two positive roots are situated between 2 and 3.

When $x=-3$ the signs of the functions are $-\ +\ -\ +$, and when $x=-2$ the signs of the functions are $+\ +\ -\ +$;

hence the negative root is between -2 and -3. To separate the two roots which lie between 2 and 3 we must substitute for x some number or numbers lying between 2 and 3. When $x = 2\frac{1}{2}$ the signs of the functions are $- \; - \; \pm \; +$. Here we have only one variation whether we consider the vanishing function $f_1(x)$ to be positive or negative; hence one of the positive roots lies between 2 and $2\frac{1}{2}$, and the other between $2\frac{1}{2}$ and 3.

2. $x^3 - 3x^2 - 12x + 24 = 0$.

Ans. Three; one between 1 and 2, one between 4 and 5, and one between -3 and -4.

3. $x^3 + 6x^2 + 10x - 1 = 0$.

4. $x^3 - 6x^2 + 8x + 40 = 0$.

5. $x^4 + 4 = 0$.

6. $x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0$.

7. $x^7 - 9x^5 + 6x^4 + 15x^3 - 12x^2 - 7x + 6 = 0$.

8. $x^4 + x^3 - x^2 - 2x + 4 = 0$.

HORNER'S METHOD OF APPROXIMATION.

615. Let it be required to find a root of the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots + Kx + L = 0 \quad . \quad . \quad . \quad (1).$$

Suppose a to be the integral part of the root required, and $r, s, t, \ldots,$ taken in order, to be the digits of the fractional part.

Let a be found by trial (**608**) or by Sturm's Theorem; then find an equation whose roots shall be less by a than those of (1) (**598**).

Let $y^n + A'y^{n-1} + B'y^{n-2} + \ldots + K'y + L' = 0 \ldots (2)$ be that equation.

In this equation the value of y is less than 1; hence the terms containing the higher powers of y are comparatively small; neglecting these, we have, approximately,

$$K'y + L' = 0, \quad \text{whence} \quad y = -\frac{L'}{K'}.$$

The first figure in the value of y is r.

Now find an equation whose roots shall be less by r than those of (2). Let $z^n + A''z^{n-1} + B''z^{n-2} + \ldots + K''z + L'' = 0 \ldots (3)$ be that equation.

In this equation the value of z is less than .1; hence, we have, approximately, $K''z + L'' = 0$; whence $z = -\frac{L''}{K''}$. This process may be continued to any desired extent, and we shall have finally $x = a + r + s + t + \ldots.$

RULE.

I. *Find the integral part of the root by Sturm's Theorem or otherwise.*

II. *Find an equation whose roots shall be less than those of the given equation by the integral part of the required root.*

III. *Divide the independent term of the transformed equation by the coefficient of the adjacent term, change the sign of the quotient and write the first figure of the result as the first figure of the fractional part of the root.*

IV. *Find an equation whose roots shall be less than those of the second equation by the first figure in the fractional part of the required root.*

V. *Divide the independent term of this transformed equation by the coefficient of the adjacent term, change the sign of the quotient, and write the first figure of the result as the second figure of the fractional part of the required root.*

VI. *Continue this process until the root is obtained to the required degree of accuracy.*

SCH. 1.—To obtain the negative roots it is best to change the signs of the alternate terms of the given equation, and then find the *positive* roots of the result; changing the signs of these, we obtain the negative roots required.

SCH. 2.—If a trial figure of the root, obtained by any division, causes the two last terms of the succeeding equation to have the same sign, that figure is not the correct one and must be changed.

SCH. 3.—If K' should reduce to zero in the operation, then we should have, approximately, $J'y^2 + L' = 0$; whence $y = \sqrt{-\frac{L'}{J'}}$.

EXAMPLES.

1. Find one root of the equation $x^3 - 2x^2 - 20x - 40 = 0.$

By Sturm's Theorem we find that this equation has only one real root, and that the integral part of this root is 6. We now find two figures of the fractional part as follows:

$$
\begin{array}{lllll}
1 & -\ 2 & -20 & -40 & \underline{|6.23} \\
 & +\ 6 & +24 & +24 & \\
\hline
 & +\ 4 & +\ 4 & -16^{(1)} & \\
 & +\ 6 & +60 & & \\
\cline{2-3}
 & +10 & +64^{(1)} & & \\
1^{(1)} & +\ 6^{(1)} & & & \\
\hline
1^{(1)} & +16^{(1)} & +64^{(1)} & -16^{(1)} & \\
 & +\ 0.2 & +33.24 & +13.448 & \\
\cline{2-4}
 & +16.2 & +67.24 & -\ \ 2.552^{(2)} & \\
 & +\ 0.2 & +\ \ 3.28 & & \\
\cline{2-3}
 & +16.4 & +70.52^{(2)} & & \\
 & +\ 0.2 & & & \\
\cline{2-2}
 & +16.6^{(2)} & & & \\
\hline
1^{(2)} & +16.6^{(2)} & +70.52^{(2)} & -\ \ 2.552^{(2)} &
\end{array}
$$

We find the coefficients of an equation whose roots are less by 6 than those of the given equation, using the method explained in Art. **598.** These coefficients are 1, 16, 64, and — 16, marked (1) in the operation. Dividing 16 by 64, we obtain .2, which is the second figure of the root. We next find the coefficients of an equation whose roots are less by .2 than those of the second equation. These coefficients are marked (2) in the operation. Dividing 2.552 by 70.52, we obtain .03, which is the third figure of the root. This process may be continued until the root is obtained to any required degree of accuracy.

2. Find one root of the equation $x^4 + x^3 - 30x^2 - 20x - 20 = 0.$

By Sturm's Theorem, we find the integral parts of the two real roots to be 5 and — 5. Changing the signs of the alternate terms of the equation, we find the fractional part of the negative root as follows:

1	-1	-30	$+20$	-20	5.73
	$+5$	$+20$	-50	-150	
	$+4$	-10	-30	$-170^{(1)}$	
	$+5$	$+45$	$+175$		
	$+9$	$+35$	$+145^{(1)}$		
	$+5$	$+70$			
	$+14$	$+105^{(1)}$			
	$+5$				
	$19^{(1)}$				
$1^{(1)}$	$+19^{(1)}$	$+105^{(1)}$	$+145^{(1)}$	$-170^{(1)}$	
	$+0.7$	$+13.79$	$+83.153$	$+159.7071$	
	$+19.7$	$+118.79$	$+228.153$	$-10.2929^{(2)}$	
	$+.7$	$+14.28$	$+93.149$		
	$+20.4$	$+133.07$	$+321.302^{(2)}$		
	$+.7$	$+14.77$			
	$+21.1$	$+147.84^{(2)}$			
	$+.7$				
	$21.8^{(2)}$				
$1^{(2)}$	$+21.8^{(2)}$	$+147.84^{(2)}$	$+321.302^{(2)}$	$-10.2929^{(2)}$	

Hence the negative root of the given equation is $-5.73+$.

Find the real roots of the following equations:

3. $x^3+2x^2-23x-70=0$. *Ans.* 5.1345.

4. $x^3-x^2+70x-300=0$. *Ans.* 3.7387.

5. $x^3+x^2-500=0$. *Ans.* 7.6172.

6. $x^3-x^2-40x+108=0$. *Ans.* $\begin{cases} 3.3792, \\ 4.5875, \\ -6.6967. \end{cases}$

7. $x^3-4x^2-24x+48=0$. *Ans.* $\begin{cases} 1.7191, \\ 6.5461, \\ -4.2652. \end{cases}$

8. $x^4+x^3+x^3-x-500=0$. *Ans.* $\begin{cases} 4.4604, \\ -4.9296. \end{cases}$

9. $x^4-9x^3-11x^2-20x+4=0$. *Ans.* $\begin{cases} .1796, \\ 10.2586. \end{cases}$

616. SYNOPSIS FOR REVIEW.

CHAPTER XXIII. THEORY OF EQUATIONS.

- DEFINITIONS
  - *The general equation of the n^{th} degree.*
  - *The absolute or independent term.*
  - *A function of a quantity.*
  - *A rational integral function of x.*
  - *A root of the equation $f(x) = 0$.*
- GENERAL PROPERTIES.
  - *When $f(x)$ is divisible by $x - r$.*
  - *When r is a root of $f(x) = 0$.*
  - *Number of roots of $f(x) = 0$.*
  - *To find an equation whose roots are given.*
  - *Relation between the coefficients of $f(x)$ and the roots of $f(x)=0$.* 1. 2. 3. *Cor. 1, 2, 3, 4, 5, 6.*
  - *When $f(x) = 0$ cannot have a root which is a rational fraction.*
  - *Roots of the form of $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$. Cor. 1, 2, 3, 4.*
- TRANSFORMATION OF EQUATIONS.
  - *To change the signs of the roots of an equation.*
  - *To transform an equation containing fractional coefficients into another in which the coefficients are integers, that of the first term being unity.*
  - *To transform an equation into another, the roots of which differ from those of the given equation by a given quantity.* Rule.
  - *To cause the second or third term of an equation to disappear. Cor.*
- THEOREM OF DESCARTES. *Cor. 1, 2, 3.*
- DERIVED FUNCTIONS
  - *Primitive function.*
  - *First derivative, Second derivative, etc.*
  - *First derivative of product of functions.*
- ROOTS COMMON TO TWO EQUATIONS. EQUAL ROOTS.
- LIMITS OF THE ROOTS OF AN EQUATION.
  - *Number of roots of $f(x) = 0$ lying between p and q. Cor. 1, 2, 3.*
  - *Limits of positive roots.*
  - *Limits of negative roots.*
- STURM'S THEOREM
  - *I, II, III, IV, V, VI.*
  - *Sch.*
- HORNER'S METHOD OF APPROXIMATION. *Rule.*